New Results on Online Replacement Problem

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Abstract. The replacement problems are extensively studied from a number of disciplines including economics, finance, operations research, and decision theory. Much of previous theoretical work is "Bayesian". In this paper, we restudy the on-line replacement problem by using the competitive analysis. The goal is to minimize the total cost of cumulative payment flow plus changeover costs. Firstly, a refusal strategy is proposed and the competitive ratio for $k = 1$ is obtained. Furthermore, a new time-independent strategy S_{new} is presented and we prove that it is r-competitive when $M \in [c, d]$. Finally, weights are introduced to the original model and some results are achieved.

1 Introduction

R.Bellman [1] studied the replacement problem via dynamic programming. This problem is described as follows: The flow rate $f(t)$ is chosen by an adversary from the real interval $[m, M]$ where $0 < m \leq M$. For a start, suppose that both m and M are known to the online player. At each time $0 < t < T$ the online player can changeover and continue paying money at the rate $f(t)$. The changeover cost is 1 unit. The player chooses any number k of changeover times. The total cost consists of payment flow and changeover costs. The objective function is

$$y = k + \sum_{i=0}^{k} (t_{i+1} - t_i) f(t_i)$$

The previous research work [5, 7] of the replacement problems is mainly based on the conventional "average case analysis" . R.E.Yaniv et.al. [2, 4, 8] first initiated the worst-case competitive analysis of the *on-line replacement* problem. Namely, the online player knows nothing about the future but decides to select a replacement strategy based on past and present information. The off-line player knows the strategy which the on-line player adopt and chooses a flow function to maximize the competitive ratio [3, 9]. In the paper [2], some results were obtained. For example, the general lower bound on the competitive ratio of any deterministic strategy were presented, which is $\Theta(\frac{lnM}{lnlnM})$ for a fixed m.

In the paper [6], Azar et.al. studied the discrete time replacement problem variant with multiple, permanent replacement options and achieved some results.

X. Deng and Y. Ye (Eds.): WINE 2005, LNCS 3828, pp. 554–563, 2005.

For the convex variant a simple 7-competitive algorithm was obtained and the competitive ratio $O(min\{log(cr_{max}), loglog(cf_{max}), log(cn_{max})\})$ was achieved to study the non-convex variant.

2 Our Contributions

In this paper, we study the on-line replacement problem P_0 and variant problem P_1. P_1 is the weighted version of P_0. Based on the time-independent strategy $S_\rho^{***}(m, M)$ [2], we present a refusal strategy which the refusal times sequence is no-increasing. Namely, the online player makes the ith changeover at time t if $f(t) \leq M_i(t)$. Otherwise, it refuses to changeover. In this case, when the changeover save costs more than the penalty costs($\frac{1}{\rho^{k-1}} times$) the online player will choose to changeover. we obtain the competitive ratio $\sqrt{\frac{M+1}{m+1}}$ for $\sqrt{\frac{M+1}{m+1}} < 2$.

Next, we propose a new time-independent strategy S_{new} in which the online player changes over for the ith time when the flow rate decreases to the level of M_i. We get the following results: The changeover sequence M_i designed strictly decreases below m within $k+1$ steps; S_{new} is r-competitive for $M \in [c, d]$ where $c = m + \sqrt{r}$ and $d = \frac{(\sqrt{r(\lfloor r \rfloor+1)} - \sqrt{\frac{\lfloor r \rfloor}{r}})m + r - \lfloor r \rfloor}{1 - \sqrt{\frac{\lfloor r \rfloor}{r}} + \sqrt{r(\lfloor r \rfloor+1)} - r}$.

Finally, we present the weighted replacement problem. The weighted value $\eta \in (0, 1)$ weights the relative importance of payment flow and changeover. It has very realistic meaning to consider this factor. For example, the replacement costs of cars are clearly different from those of computers and this will affect the choice of the replacement value in different phase. The introduction of this parameter does lead our mathematic model further to approach the real life. In this section, we propose a characterization theorem that gives necessary and sufficient conditions. This characterization theorem provides an efficient tool for determining the competitive ratio. In the rest of this section, we construct a refusal strategy $WS(m, M)$ and obtain the following results: The changeover time is $k = \lceil(\beta m + 1)(r - 1)\rceil$. If $\sqrt{\frac{\beta M}{\beta m+1}} \leq \frac{\beta m+2}{\beta m+1}$, then $r = \sqrt{\frac{\beta M}{\beta m+1}}$ and $k = 1$.

3 Replacement Problem P_0

Let $\{M_i\}_{i=1}^k$ be the changeover thresholds sequence which is strictly decreasing within the open interval (M, m) and $\{b_i\}_{i=1}^k$ be the refusal times sequence which is non-increasing with the time horizon $[0, 1]$.

3.1 A Refusal Strategy Based on $S_\rho^{***}(m, M)$

In the paper [3], a time-independent policy $S_\rho^{***}(m, M)$ with the changeover thresholds sequence $\{M_i\}_{i=1}^k$, has been presented, which shows that the sequence defined by this strategy decreases below m within $k+1$ steps for sufficiently large ρ. We quote as follows:

The sequence of changeover thresholds $\{M_i\}$ is

$$\begin{cases} M_0 &= M \\ M_{i+1} &= \frac{M_i+k}{\rho} - 1, \qquad \text{integer } i \geq 1 \end{cases} \tag{1}$$

where each $\rho > 1$, and $k = \lfloor \rho \rfloor$.

This is an approximately optimal policy for $m > 0$ which has a stronger upper bound for the general case. Therefore, we propose a new refusal policy based on above sequence of changeover thresholds. In this section, we consider the refusal strategy S_r, which is the time-independent strategy $S_\rho^{***}(m, M)$ with refusal time.

3.1.1 Competitive Analysis of S_r When $k = 1$

It pays the penalty ($\frac{1}{\rho^{k-1}} times$) the online player should changeover. We present a refusal times sequence $\{b_i\}$ as follows.

$$b_i = \begin{cases} 1 - \frac{1}{\rho^{k-1}(M_{i-1}-M_i)} & 0 \leq i \leq k \\ 0 & i = k+1 \end{cases} \tag{2}$$

Lemma 1. *For all* $1 \leq i \leq k$, $\{b_i\}$ *is non-increasing.*

Proof. From (1), we can get $M_j = \frac{M}{\rho^j} + \frac{\rho-k}{(\rho-1)\rho^j} + \frac{k-\rho}{\rho-1}$ by induction on j. Therefore, we obtain

$$\begin{aligned} M_{i-1} - M_i &= \frac{M}{\rho^{i-1}} + \frac{\rho-k}{(\rho-1)\rho^{i-1}} + \frac{k-\rho}{\rho-1} - \frac{M}{\rho^i} - \frac{\rho-k}{(\rho-1)\rho^i} - \frac{k-\rho}{\rho-1} \\ &= (M + \frac{\rho-k}{\rho-1})(\frac{1}{\rho^{i-1}} - \frac{1}{\rho^i}) \\ &= \frac{M(\rho-1)+\rho-k}{\rho^i} \end{aligned} \tag{3}$$

For every $\rho > 1$, $M_{i-1} - M_i$ is strictly decreasing with i. Hence, substitute equality (2) with (3), we get the following result:

$$b_i = 1 - \frac{\rho^{i-k+1}}{M(\rho-1)+\rho-k}$$

It is not hard to see that $\{b_i\}$ is non-increasing with i. □

Lemma 2. *For all* $1 \leq i \leq k$, $b_i \in [0, 1]$.

Proof. From (3), we obtain

$$\rho^{k-1}(M_{i-1} - M_i) = \rho^{k-1-i}(M(\rho-1)+\rho-k) \tag{4}$$

We can observe that

$$\rho^{k-i-1} \geq \rho^{-1} \tag{5}$$

for $k \geq i$.

From the assumption $M > \frac{\lfloor\rho\rfloor}{\rho-1}$, it implies that

$$M(\rho - 1) + \rho - k > \rho \tag{6}$$

Hence, substitute (4) with (5) and (6), we obtain

$$\rho^{k-1}(M_{i-1} - M_i) > 1 \tag{7}$$

It clearly follows that $b_i \in [0, 1]$ from (2) and (7). □

3.1.2 Competitive Analysis of S_r When $k = 1$

Theorem 1. *The competitive ratio of* S_r *is* $\sqrt{\frac{M+1}{m+1}}$ *for* $\sqrt{\frac{M+1}{m+1}} < 2$*.*

Proof. When $k = 1$, from (1) and (2), we can write S_r as follows:

$$M_1 = \frac{M+1}{r} - 1 \tag{8}$$

$$\begin{aligned} b_1 &= 1 - \frac{1}{\rho^0(M - M_1)} \\ &= 1 - \frac{1}{M + 1 - \sqrt{m+1}M + 1} \end{aligned} \tag{9}$$

$$M_2 = \frac{M_1 + 1}{r} - 1 = \frac{M+1}{r^2} - 1 \leq m \tag{10}$$

Solving the inequality (10), we obtain $r \geq \sqrt{\frac{M+1}{m+1}}$. We know that $r < 2$ for the assumption of $k = \lfloor r \rfloor$, hence $\sqrt{\frac{M+1}{m+1}} < 2$.

In the rest of this section, we will claim that the refusal strategy S_r will achieve the competitive ratio r. Like the paper [3], for $b_2 = 0$, the condition C_1 can be rewrite as follows:

$$M \leq r \cdot min\{M, M_1 + 1, Mb_1 + m(1 - b_1) + 1, M_1b_1 + m(1 - b_1) + 2\} \tag{11}$$
$$M_1 + 1 \leq r \cdot min\{M, M_1 + 1, m + 1, m + 2\} \tag{12}$$
$$M_1 + 1 \leq r \cdot min\{M, m + 1, m + 2\} \tag{13}$$

We know that $M > m + 1$ and $M_1 > m$. Clearly, inequalities (12) and (13) hold for $m + 1$ is the minimum of the righthand side.

Next, we will check inequalities (11). The following four cases should be considered.

Case 1. $M \leq rM$, it's trivially holds for $r \geq 1$.

Case 2. $M \leq r(M_1 + 1) = M + 1$ holds.

Case 3. $M \leq r(Mb_1 + m(1 - b_1) + 1)$. Substituting r with $\sqrt{\frac{M+1}{m+1}}$ and b_1 with (10), we obtain following result.

$$\begin{aligned} r(Mb_1 + m(1 - b_1) + 1) &= r(M(1 - \frac{1}{M - M_1}) + m\frac{1}{M - M_1} + 1) \\ &\geq r(M + 1 + \frac{\frac{M+1}{r^2} - 1 - M}{M - \frac{M+1}{r} + 1}) \\ &= rM - 1 \end{aligned} \tag{14}$$

We note that the assumption $M > \frac{k}{r-1}(k = 1$ in this case), which implies that $rM - 1 > M$. Hence, this inequality holds.

Case 4. $M \leq r \cdot (M_1 b_1 + m(1 - b_1) + 2)$. We can see that

$$\begin{aligned} r \cdot (M_1 b_1 + m(1 - b_1) + 2) &= r(M_1(1 - \frac{1}{M - M_1}) + (\frac{M+1}{r^2} - 1)\frac{1}{M - M_1} + 2) \\ &= r(M_1 + 2) - 1 \\ &= r(M + 1) \end{aligned} \tag{15}$$

Therefore this inequality holds for $r \geq 1$.

Finally, For $b_1 = 1 - \frac{1}{M - M_1}$ we can prove that $Mb_1 + M_1(1 - b_1) + 1 = M$. It is easy to see that condition C_2 is identical to *Case* 4, and condition C_2 holds. □

3.2 A New Optimal Strategy

In this section, we propose another time-independent strategy S_{new} which can obtain the competitive ratio r when M is an element of the real interval.

Let $\{M_i\}_{i=1}^{k}$ be the changeover thresholds sequence. Then we define

$$\begin{cases} M_0 = M \\ M_i = M - \sqrt{\frac{i}{r}}(M - m) \qquad \text{integer } i \geq 1 \end{cases} \tag{16}$$

where $r > 1$ and $k = \lfloor r \rfloor$

Lemma 3. *In this case, the sequence $\{M_i\}$ strictly decreases below m within $k + 1$ steps.*

Proof. For $k = \lfloor r \rfloor$, we obtain that $r < k + 1 \leq r + 1$. Set $i = k + 1$, and we get the following result from (16).

$$\begin{aligned} M_{k+1} &= M - \sqrt{\frac{k+1}{r}}(M - m) \\ &\leq m \end{aligned} \tag{17}$$

□

Theorem 2. *For $M \in [c, d]$, S_{new} is r-competitive.*

Proof. Set $b_{k+1} = 0$ and $b_i = 1, i = 1, 2, \cdots, k$. The two sufficient and necessary conditions of the paper [2] reduce to the following condition.
C_1 *for all* $0 \leq i \leq j \leq k$,

$$M_i + j \leq r \cdot min\{M, M_{i+1} + 1\} \tag{18}$$

Set $c = m + \sqrt{r}$, and $d = \frac{(\sqrt{r(\lfloor r \rfloor + 1)} - \sqrt{\frac{\lfloor r \rfloor}{r}})m + r - \lfloor r \rfloor}{1 - \sqrt{\frac{\lfloor r \rfloor}{r}} + \sqrt{r(\lfloor r \rfloor + 1)} - r}$.

For $M \geq c$, we can get

$$\begin{aligned} M &\geq m + \sqrt{r} \\ &= M - \sqrt{\frac{1}{r}}(M - m) + 1 \\ &= M_1 + 1 \end{aligned} \tag{19}$$

Hence we only confirm that $M_i + j \leq r \cdot (M_{i+1} + 1)$ holds.

For $M \leq d$, and substituting $\lfloor r \rfloor$ with k, we can obtain the following result.

$$M - \sqrt{\frac{k}{r}}(M - m) + k \leq rM - \sqrt{r(k+1)}(M - m) + r \tag{20}$$

For $i \in [1, k]$, inequality (20) can be expressed as follows.

$$M - \sqrt{\frac{i}{r}}(M - m) + k \leq rM - \sqrt{r(i+1)}(M - m) + r$$

Namely, $M_i + k \leq r(M_{i+1} + 1)$. Hence, condition C_1 holds and S_{new} is r-competitive. □

4 Weighted Replacement Problem P_1

In this section, we consider a class of the weighted replacement problem. As discussed in the introduction, to model this problem, we select an objective function to trade off payment flow and changeover costs. The factor $\eta \in (0, 1)$ weights the relative importance of payment flows and changeover. Compared with original problem P_0, the objective function of P_1 is defined as follows:

$$y^{'} = \eta k + (1 - \eta) \sum_{i=0}^{k} (t_{i+1} - t_i) f(t_i) \tag{21}$$

where k is the changeover times, $\eta \in (0, 1)$ is the factor that weights the cost of payment flow versus the cost of changeover.

Without loss of generation, let $\beta = \frac{1-\eta}{\eta}$ and the equation (21) can be expressed:

$$y = k + \beta \sum_{i=0}^{k} (t_{i+1} - t_i) f(t_i) \tag{22}$$

We present a characterization theorem which can help to establish a refusal strategy and achieve a competitive ratio r.

Lemma 4. *S is r-competitive if and only if the following two conditions hold:* C_1 *for* $0 \le i \le j \le k$,

$$\beta M_i b_{j+1} + \beta M_j (1 - b_{j+1}) + j \le r \cdot Min \begin{bmatrix} \beta M_0 \\ \beta M_{i+1} + 1 \\ \beta M_0 b_{j+1} + \beta m(1 - b_{j+1}) + 1 \\ \beta M_{i+1} b_{j+1} + \beta m(1 - b_{j+1}) + 2 \end{bmatrix}$$

C_2 *for* $0 \le i < j \le k$

$$\beta M_i b_j + \beta M_j (1 - b_j) + j \le r \cdot Min \begin{bmatrix} \beta M_0 \\ \beta M_{i+1} + 1 \\ \beta M_0 b_j + \beta m(1 - b_j) + 1 \\ \beta M_{i+1} b_j + \beta m(1 - b_j) + 2 \end{bmatrix}$$

Proof. The proof is similar to the proof of the paper [3] presented. Hence we omit it here. □

4.1 The Refusal Strategy $WS(m, M)$

In this section, we construct a refusal strategy $WS(m, M)$ and analyze its competitive ratio. The sequence of changeover thresholds $\{M_i\}$ is

$$\begin{cases} M_0 &= M \\ M_{i+1} &= \frac{\beta M_i + i}{\beta r} - \frac{1}{\beta}, \qquad 0 \le i < k \end{cases} \tag{23}$$

where each $r > 1$.

And the refusal times sequence $\{b_i\}$ is

$$b_i = \begin{cases} 1 - \frac{1}{\beta(M_{i-1} - M_i)}, & 0 \le i \le k \\ 0, & i = k + 1 \end{cases} \tag{24}$$

In the rest of this section, we will claim $b_i \in (0, 1)$. Firstly, we obtain M_i by induction on j.

$$M_i = (M_0 + \frac{r^2}{\beta(r-1)^2})r^{-i} - \frac{r^2}{\beta(r-1)^2} + \frac{i}{\beta(r-1)}$$

Therefore, set $\Delta = M_0 + \frac{r^2}{\beta(r-1)^2}$, and we can get following result:

$$\begin{aligned} M_{i-1} - M_i &= \frac{\Delta}{r^{i-1}} - \frac{r^2}{\beta(r-1)^2} + \frac{i-1}{\beta(r-1)} - \frac{\Delta}{r^i} + \frac{r^2}{\beta(r-1)^2} - \frac{i}{\beta(r-1)} \\ &= \frac{\beta M_0 (r-1)^2 + r^2 - r^i}{\beta(r-1) r^i} \end{aligned} \tag{25}$$

which implies that the difference $M_{i-1} - M_i$ is decreasing with i.

Using the equations $M_{k-1} = \frac{r(\beta M_k+1)-k+1}{\beta}$ and $M_k = \frac{r(\beta m+1)-k}{\beta}$, we get

$$\begin{aligned} M_{k-1} - M_k &= \frac{r(\beta M_k + 1) - k + 1}{\beta} - \frac{\beta M_k}{\beta} \\ &= (r-1)M_k + \frac{r-k+1}{\beta} \\ &= \frac{(r-1)(r(m\beta+1)-k)}{\beta} + \frac{r-k+1}{\beta} \end{aligned} \tag{26}$$

It is easy to see that

$$k < r(\beta m{+}1) - \beta m \tag{27}$$

for $M_{k+1} = m$ and $M_k > m$.

Substituting inequation (25) into equation (24), we achieve

$$\begin{aligned} M_{k-1} - M_k &= \frac{(r-1)(r(m\beta+1)-k)}{\beta} + \frac{r-k+1}{\beta} \\ &> \frac{1}{\beta} \end{aligned} \tag{28}$$

Therefore, we can get following Lemma from (22) and (26).

Lemma 5. *For* $0 \le i \le k$, $b_i \in (0,1)$.

We know that M_{k+1} is the minimal value, so $M_{k+1} = m \le M_{k+2} = \frac{\beta m+k+1}{\beta r} - \frac{1}{\beta}$. We can obtain

$$k \ge r(\beta m + 1) - \beta m - 1 \tag{29}$$

From (27) and (29), we achieve the following result.

Lemma 6. *The changeover time is* $k = \lceil(\beta m + 1)(r-1)\rceil$.

Theorem 3. *If* $\sqrt{\frac{\beta M}{\beta m+1}} \le \frac{\beta m+2}{\beta m+1}$*, then* $r = \sqrt{\frac{\beta M}{\beta m+1}}$ *and* $k = 1$.

Proof. If $\sqrt{\frac{\beta M}{\beta m+1}} \le \frac{\beta m+2}{\beta m+1}$, then we obtain $k \le 1$ from Lemma 6. Namely, WS strategy consists of one changeover threshold. Set $r = \sqrt{\frac{\beta M}{\beta m+1}}$, we can get following result from (23).

$$\begin{aligned} M_1 &= \frac{\beta M}{\beta r} - \frac{1}{\beta} \\ &= \frac{\sqrt{\beta M}\sqrt{\beta m+1} - 1}{\beta} \end{aligned} \tag{30}$$

And

$$\begin{aligned} b_1 &= 1 - \frac{1}{\beta M + 1 - \sqrt{\beta M}\sqrt{\beta m+1}} \\ &= \frac{\beta M - \sqrt{\beta M(\beta m+1)}}{\beta M + 1 - \sqrt{\beta M(\beta m+1)}} \end{aligned} \tag{31}$$

We know that b_2 is set to zero. Therefore condition C_1 reduces to the following equalities from Lemma (5).

$$\beta M \leq r \cdot min\{\beta M, \beta M_1+1, \beta M b_1 + \beta m(1-b_1)+1, \beta M_1 b_1 + \beta m(1-b_1)+2\} \tag{32}$$

$$\beta M_1 + 1 \leq r \cdot min\{\beta M, \beta M_1 + 1, \beta m + 1, \beta m + 2\} \tag{33}$$

$$\beta M_1 + 1 \leq r \cdot min\{\beta M, \beta m + 1, \beta m + 2\} \tag{34}$$

It is easy to see that $min\{\beta M, \beta M_1 + 1, \beta m + 1, \beta m + 2\} = \beta m + 1$ for $\beta M > \beta m + 1$. Otherwise, the on-line and the off-line player never changeover and the competitive ratio is 1. We only verify $\beta M_1 + 1 \leq r(\beta m + 1)$ for inequality (33) and (34). It holds for $M_2 = m$.

Next we consider the inequality (32) which has four cases discussed.

Case 1 and *Case* 2 clearly hold. By setting $i = 0$ the result can be obtained from (23).

Considering *Case* 3, and substitute b_1 with (31) and r with $\sqrt{\frac{\beta M}{\beta m+1}}$ we will confirm the following inequality holds.

$$\beta(M - M_1)^2 \geq M - m \tag{35}$$

For $M_1 = \frac{\sqrt{\beta M}\sqrt{\beta m+1}-1}{\beta}$ and $\sqrt{\frac{\beta M}{\beta m+1}} \leq \frac{\beta m+2}{\beta m+1}$, so we need to prove

$$\begin{aligned}\beta(M - M_1)^2 &\geq \frac{(\beta(M-m)-1)^2}{\beta} \\ &\geq M - m\end{aligned} \tag{36}$$

We know that $\beta M \geq \beta m + 1$, hence inequality (36) holds.

Case 4, we need to check that the following inequality holds.

$$\beta M \leq r(\beta M_1 b_1 + \beta m(1 - b_1) + 2) \tag{37}$$

Similar to *Case* 3, we substitute b_1 and r again, and to verify the inequality (37) holds we need to prove $M - M_1 \geq M_1 - m$. This inequality obviously holds.

Finally, for $0 \leq i < j \leq 1$, condition C_2 is $\beta M b_1 + \beta M_1(1 - b_1) + 1 \leq r \cdot min\{\beta M, \beta M_1 + 1, \beta M b_1 + \beta m(1 - b_1) + 1, \beta M_1 b_1 + \beta m(1 - b_1) + 2\}$. It is easy to see that $\beta M b_1 + \beta M_1(1 - b_1) + 1 = \beta M$ for substituting b_1 with $1 - \frac{1}{\beta(M-M_1)}$. Therefore condition C_2 holds. □

5 Concluding Remarks

In this paper, we restudy the online replacement problem and introduce the weight in the original model. Although this is only one step toward a more realistic solution of the problem, the introduction of this parameter considerably complicates the analysis and achieves some different results. There are some possible extensions that will lead towards more realistic models. For example, because the replacement problem is an important area of the financial decision, the methods of measurement used must be based on the concept of "time value of money". These include the methods of discount rate of return(DRR), net present value(NPV) and so on. Another research direction is how to introduce the risk-reward model [10] to this problem.

Acknowledgements

The work was partly supported by the National Natural Science Foundation of China (No.70471035,10371094,70401006).

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