On Stable Line Segments in Triangulations [1]

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1 Overview

Let S be a set of n points in the plane and E denote the set of all the line segments with endpoints in S. A line segment $\overline{pq}$ with $p, q \in S$ is called a **stable line segment** of all triangulations of S, if no line segment in E properly intersects $\overline{pq}$. The intersection of all possible triangulations of S then is the set of all stable line segments in S, denoted by $SL(S)$.

As a combinatorial problem, various properties of stable line segments of a set of planar points have been investigated in [13]. It is shown that the maximum number of stable line segments in S is $2(n-1)$. There is an interesting relationship between stable line segments and so-called extreme line segments $EL(S)$ [6]. A line segment $\overline{pq}$ with $p, q \epsilon S$ is called an extreme line segment if $\{p, q\} = E \cap H$ for some open half-plane H [6]. Then, we have that

$$CH(S) \subseteq EL(S) \subseteq SL(S).$$

A more important property is the relationship between $SL(S)$ and so-called k-optimal triangulations. Let $T(S)$ denote a triangulation of S. $T(S)$ is called a **k-optimal triangu-**

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lation for $4 \leq k < n$, denoted by $LOT_k(S)$, if every k-sided simple polygon drawn from $T(S)$ is optimally triangulated by some edges of $T(S)$.

Let $SL_k(S)$ denote the intersection of all possible $LOT_k(S)$'s (i.e., the set of edges that are in every $LOT_k(S)$). Let $MWT(S)$ denote a minimum weight triangulation of S. Then, we have that

$$SL(S) \subseteq SL_4(S) \subseteq \cdots \subseteq SL_k(S) \cdots \subseteq SL_{n-1}(S) \subseteq MWT(S).$$

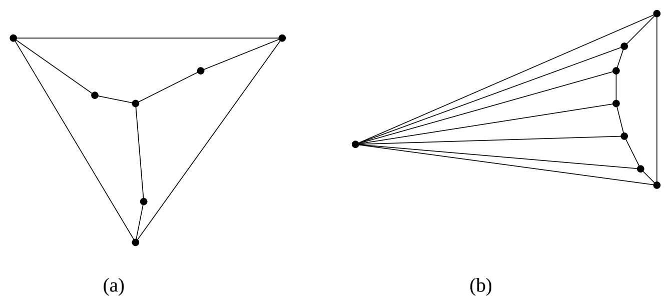

Figure 1:

In some special cases of S, $SL(S)$ forms a connected graph as shown in Figure 1. Thus, an $MWT(S)$ can be constructed in polynomial time using the dynamic programming algorithm proposed in [7, 10].

So far the structure properties of $SL(S)$ have been thoroughly studied, but not its algorithmic issue.

A recent result on finding a subgraph $LOT(S)$ of $SL_4(S)$ [5] implies an $O(n^4)$ time and $O(n^3)$ space algorithm for finding $SL(S)$ since it is not difficult to show that

$$SL(S) \subseteq LOT(S) \subseteq SL_4(S).$$

In this paper, we shall propose two algorithms for computing $SL(S)$. One is an $O(n^2 \log n)$ time and $O(n)$ space algorithm and the other is an $O(n^2)$ time and $O(n^2)$ space algorithm.

2 Introduction

A triangulation of a planar point set S is defined as a maximal set of non-crossing line segments which have both endpoints in S. A minimum weight triangulation of S (denoted $MWT(S)$) is a triangulation among all possible triangulations over S such that the sum of its total edge lengths is minimal. To compute an MWT of a point set is an outstanding open problem, whose complexity status is unknown since 1975 [12, 8]. An $O(n^3)$ time dynamic programming algorithm for constructing an MWT of a simply polygon was given independently in [7, 10]. Based on the above mentioned dynamic programming algorithm, Anagnostou and Corneil [1] designed an $O(n^{3k+1})$ time algorithm for computing an MWT of a point set with k nested convex polygons, and later Meijer and Rappaport [11] improved the time complexity to $O(n^k)$ when each of the k nested polygons degenerated into a straight line segment. Xu and others [13, 3] showed that if a subgraph of an MWT with k connected components is given, then an MWT can be found in $O(n^{k+2})$ time. Up to now, none of the existing algorithms for finding an MWT of a general point set achieves polynomial time bound. An alternative direction is to identify a subset of line segments in E belonging to an MWT. The advantage of this direction is two-fold. The more such line segments are identified, the more likely the resulting subgraph will connect all the points in S. Then, the ultimate solution can be found in $O(n^{k+2})$ time by using dynamic programming. On the other hand, it was shown in [15] that finding more line segments within an MWT can improve the performance of some heuristics.

Several investigations have reported on the subgraphs of MWT [2, 4, 5, 9, 13, 14, 16]. A trivial case is the set of line segments in all triangulations of a given point set S (i.e., a set of stable line segments $SL(S)$). No detailed work was done on the algorithms for computing $SL(S)$. In the following section, we shall propose two algorithms for computing $SL(S)$.

3 Algorithmic Issues

Let J denote the set of all triangulations of a point set S, then we have the following obvious facts:

Fact 1. $SL(S) = \cap_{T(S)\in J} T(S)$, and

Fact 2. $\overline{pq} \in SL(S)$ **iff** no line segment with endpoints in S properly intersects $\overline{pq}$.

Note that the Delaunay triangulation of S, $DT(S)$, belongs to J. By Fact 1, we first construct the Delaunay triangulation $DT(S)$ and then test whether the line segments in $DT(S)$ are also in $SL(S)$. Note that the number of line segments in $DT(S)$ is linearly proportional to n, it is easy to design an $O(n^3)$ time algorithm by testing all possible intersections of the line segments with Delaunay edges.

With a more detailed geometric analysis, we can improve the time complexity from $O(n^3)$ to $O(n^2 \log n)$ and space complexity from $O(n^2)$ to $O(n)$ or time complexity to $O(n^2)$ and space complexity remains as $O(n^2)$.

3.1 Algorithm 1

Lemma 1 *Let $\overline{pq}$ be a line segment, $\{p, q\} \cup S$ be a simple point set, $|S| = n$. To determine whether there is a line segment with two endpoints in S that properly intersects $\overline{pq}$ can be answered in $O(n \log n)$ time and $O(n)$ space.*

Proof First, by a rigid motion we can transform point p to the origin and point q on the x-axis and denote its coordinates $(x^*, 0), x^* > 0$. This can be done in $O(n)$ time. In the new coordinate system, S becomes S', $p \to p'$ and $q \to q'$, $p' = (0,0)$ and $q' = (x^*, 0)$, and $r = (x(r), y(r))$ in S'. If no points in S' are below (or above) x-axis, then no line segment with two endpoints in S' intersects the line segment $L(p', q')$. If there are points with $y(p_i) > 0$ and $y(p_j) < 0$ for $p_i, p_j \in S'$, we divide S' into two subsets

$$S'_+ = \{p \mid y(p) > 0, p \in S'\}$$
$$S'_- = \{p \mid y(p) < 0, p \in S'\}$$

This step can be done in $O(n)$ time.

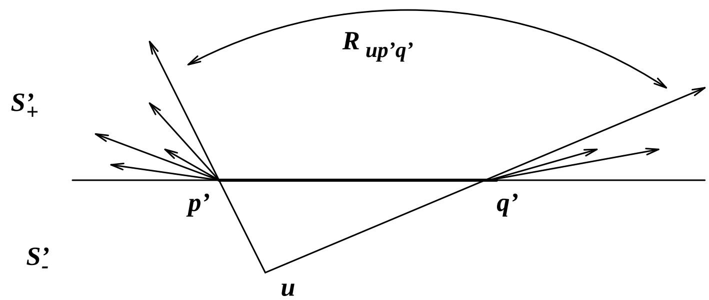

Figure 2:

We sort points in S'_+ lexicographically by polar angle at p' and q' respectively. In the new sorted polar coordinate system, S'_+ becomes $S'_+(p')$ and $S'_+(q')$ respectively. Let $\mid S'_+ \mid= m$, and $\alpha_{p'}(r)$ denote the polar angle of ray r from origin p' and $\alpha_{q'}(r)$ denote the polar angle of r from q'. We have

$$S'_+(p') = \{p_i^+ \mid \alpha_{p'}(p_{i+1}^+) > \alpha_{p'}(p_i^+), i = 1, 2, \cdots, m-1\}$$
$$S'_+(q') = \{q_i^+ \mid \alpha_{q'}(q_{i+1}^+) > \alpha_{q'}(q_i^+), i = 1, 2, \cdots, m-1\}$$

The above sorting step can be done in $O(n \log n)$ time [PS85]. Let $u \in S'_-$. Now we consider whether there is a line segment with one endpoint u and another endpoint in S'_+ that crosses $\overline{pq}$ as follows.

Construct two rays up' and uq', let $\alpha_{up'}$ and $\alpha_{uq'}$ be the polar angles of up' and uq' in polar coordinate system with anchor points p' and q' respectively. Testing the rank of $\alpha_{up'}$ in

$S'_+(p')$ and $\alpha_{uq'}$ in $S'_+(q')$, can be done in $O(\log n)$ time by binary search. This way we can find out whether there exists a point in S'_+ lying in the angle region $R_{up'q'}$ between the two rays up' and uq'. This follows from the following simple observation. There exists a point $v \in S'_+$ such that $\overline{uv}$ crosses $\overline{pq}$ iff $rank(uq') + \mid L_+(uq') \mid < rank(up')$, where $rank(up')$ is the number of points in S'_+ with polar angle less than $\alpha_{up'}$, $rank(uq')$ is the number of points in S'_+ with polar angle less than $\alpha_{uq'}$, and $L_+(uq')$ is the set of points in S'_+ that are collinear with uq'. (See Figure 2.)

The above discussion shows that the total computation to determine whether a line segment with two endpoints in S intersects $\overline{pq}$ take at most $O(n \log n)$ time and $O(n)$ space.

□

In what follows, $LI(S, \overline{pq})$ denotes the above algorithm that answers whether or not there exists a line segment in E that crosses $\overline{pq}$. By the above lemma, algorithm $LI(S, \overline{pq})$ takes $O(n \log n)$ time and $O(n)$ space. Now we can state the theorem.

Theorem 1 *$SL(S)$ can be found in $O(n^2 \log n)$ time and $O(n)$ space, where $\mid S \mid = n$.*

Proof It is clear that $SL(S)$ must be contained in the Delaunay triangulation $DT(S)$. Thus, we start with $DT(S)$, which can be constructed in $O(n \log n)$ time and $O(n)$ space. Using algorithm $LI(S, \overline{pq})$ we test if an edge $\overline{pq}$ of $DT(S)$ belongs to $SL(S)$ in $O(n \log n)$ time and $O(n)$ space. The theorem follows since the number of edges in $DT(S)$ is $O(n)$.

□

3.2 Algorithm 2

The above time complexity bound can be reduced to $O(n^2)$ if the space bound increases to $O(n^2)$.

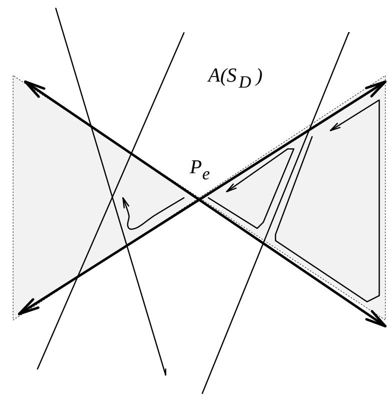

Figure 3:

Algorithm 2

- Find the arrangement for n lines, where each line is the dual of a point of S in the dual plane. Denote this arrangement as $A(S_D)$.
- Find $DT(S)$; For each Delaunay edge e of $DT(S)$ DO.
  - Let p_e be the intersection point of the dual lines of the endpoints of e. Let $W(p_e)$ be the double wedge determined by these two dual lines. Traverse the portion of $A(S_D)$ inside $W(p_e)$, starting at p_e. (Refer to Figure 3.)
  - If a vertex of $A(S_D)$ is found properly inside $W(p_e)$, then report 'e is not in SL';
  - Otherwise, report 'e is in SL'
- EndDo.

Theorem 2 *$SL(S)$ can be found in $O(n^2)$ time and $O(n^2)$ space, where $\mid S \mid = n$.*

4 Concluding Remarks

We proposed two algorithms to compute $SL(S)$; the first takes $O(n^2 \log n)$ time and $O(n)$ space, and the second takes $O(n^2)$ time and $O(n^2)$ space. It is interesting to find out whether $SL_4(S)$ can be computed in polynomial time.

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