Triangulating a Convex Polygon with Small Number of Non-standard Bars

Extended Abstract

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Abstract. For a given convex polygon with inner angle no less than $\frac{2}{3}\pi$ and boundary edge bounded by $[l, \alpha l]$ for $1 \leq \alpha \leq 1.4$, where l is a given standard bar's length, we investigate the problem of triangulating the polygon using some Steiner points such that (i) the length of each edge in triangulation is bounded by $[\beta l, 2l]$, where β is a given constant and meets $0 < \beta < \frac{1}{2}$, and (ii) the number of non-standard bars in the triangulation is minimum. This problem is motivated by practical applications and has not been studied previously. In this paper, we present a heuristic to solve the above problem, which is based on the heuristic to generate a triangular mesh with more number of standard bars and shorter maximal edge length, and a process to make the length of each edge lower bounded. Our procedure is simple and easily implemented for this problem, and we prove that it has good performance guaranteed.

1 Introduction

Generating triangular meshes is one of the fundamental problems in computational geometry, and has been extensively studied; see e.g. the survey article by Bern and Eppstein[3]. From the view point of applications, it is important to impose geometric constraints on the shape of triangles in the obtained triangulation. Several measures of triangle quality, along with various algorithms to find optimal or near-optimal triangular meshes, have been reported [1, 2, 4–6, 10, 11].

For a given length l, we say that an edge is *standard bar* if its length is l while an edge is *non-standard bar* if its length is not. In this paper, we consider the problem of generating an edge bounded triangular mesh for a given convex polygon using some Steiner points so that the number of non-standard bars in the triangulation is minimized. The problem is similar to the one that finds an edge bounded triangulation where the number of standard bars is maximized, since a triangulation that achieves one of these objectives also does it well for

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the other, i.e., if a triangulation has increased the number of standard bars, it must decrease the number of non-standard bars, and vice versa.

This problem will be formalized as follows: we are given a convex polygon $\boldsymbol{P}$ with n vertices and a standard bar length l. It is assumed that every inner angle of $\boldsymbol{P}$ is no less than $\frac{2}{3}\pi$ and the length of every boundary edge is in the interval $[l, \alpha l]$, where $1 \leq \alpha \leq 1.4$. The objective is to generate a triangulation of $\boldsymbol{P}$ with every edge length is between βl and $2l$, and in a way that the number of non-standard bars is minimized (where β is a given constant and meets $0 < \beta < \frac{1}{2}$).

To the knowledge of the authors, the problem dealt with in the present paper has not been studied in the field of computational geometry. However, this problem appears in many practical applications. For example, in architecture design where the material is limited, to triangulate a convex polygon with some standard bars and less number of non-standard bars is often considered. The standard bar can be reused for many times, but the non-standard bars can't. Furthermore, from the practical point of view, there are also some constraints for the non-standard bars, for example, the length of the non-standard bar should be neither too long nor too short compared with the standard bar.

In this paper, we present a heuristic for constructing such a triangular mesh which is similar in simplicity and efficiency to standard algorithms for triangular mesh generation. The main idea is based upon the procedure to generate a triangulation with the number of standard bars as many as possible while the maximum edge length is short, and then upon the procedure to make every edge length bounded from below by a certain length. Our heuristic is capable of producing a triangulation with each edge bounded by $\left[\beta l, \max\{l+2\beta l, \frac{\sqrt{219}}{10}l+\beta l\}\right]$, which is contained in $[\beta l, 2l]$, and the number of non-standard bars is upper-bounded by $n + \left\lceil \frac{2}{\sqrt{3}}\alpha n \right\rceil$. Note that the number of interior Steiner points and triangles can go up to $O(n^2)$, so this $O(n)$ non-standard bars introduced by our heuristic are not large in number.

The rest of this paper is organized as follows. In section 2 we first provide a heuristic to obtain a triangulation $\mathcal{M}$ such that the number of standard bars in $\mathcal{M}$ is as many as possible, and that the maximum edge length in $\mathcal{M}$ is short. We examine the triangulation $\mathcal{M}$ in great detail. Especially, we find that the upper bound of each edge length is $\frac{\sqrt{219}}{10}l$, which is a tight bound, but the lower bound is not guaranteed. In section 3 we use an approach to make each edge length bounded from below by βl. Thus the "*new*" triangulation will meet the constraints of the problem. Finally the number of non-standard bars will be investigated in section 4 and section 5 gives some future works related to this paper.

2 A Triangulation with More Number of Standard Bars and Shorter Maximal Edge Length

In this section, we consider the problem of generating a triangulation for $\boldsymbol{P}$ with the number of standard bars maximized and the length of maximal edge in the triangulation minimized. We shall give a heuristic for this problem and then show that the triangulation produced by our heuristic can be modified to give a good solution for the problem addressed in section 1.

The key idea behind the heuristic is to use the MinMax triangulation for a polygon. A MinMax edge triangulation stands for the triangulation that minimizes the maximum edge length in a triangulation over all possible triangulations of the given polygon.

Heuristic A

Step 1: Put $\boldsymbol{P}$ on the plane which is full of equilateral triangle lattice with edge length l.

Step 2: Let P' be the lattice set inside $\boldsymbol{P}$. Compute $B(P')$, where $B(P')$ denotes the boundary with lattice edges of P'.

Step 3: Let $CH(P)$ be the boundary of $\boldsymbol{P}$. Use P and $B(P')$ to triangulate the polygon region between $CH(P)$ and $B(P')$ under the MinMax edge criteria.

Let $\mathcal{M}$ be the triangulation obtained by the Heuristic A. Our aim is to present an upper bound of edge length in $\mathcal{M}$. To this end, firstly it is worth noting that, while using the Step 3 to obtain the MinMax edge triangulation, we must connect each vertex in $\boldsymbol{P}$ with its nearest vertex in $B(P')$ otherwise the maximal edge length will be longer. Thus, we define a polygon $\mathcal{A}$, which is a subgraph of $\mathcal{M}$, as follows:

Definition 1. *Let $e = (p, q)$ be a boundary edge of $\boldsymbol{P}$. Let p_1 and q_1 respectively, denote the lattice vertices nearest to p and q in $B(P')$. As polygon $\boldsymbol{P}$ is convex, pp_1 and qq_1 are on the same side of pq. We use the notation $\mathcal{A}$ to stand for the polygon composed of pq, pp_1, qq_1 and the path of lattice edges on $B(P')$ from p_1 to q_1.*

Polygon $\mathcal{A}$ may not be convex, we can not use the dynamic programming [8, 9] to obtain the MinMax edge triangulation of $\mathcal{A}$ in theory. However, as we will prove the number of edges in $\mathcal{A}$ is at most 6 in Lemma 4, the MinMax edge triangulation of $\mathcal{A}$ can be easily generated in practice.

From the above discussion, we can obtain the following lemma.

Lemma 1. *The maximum of the maximal edge length in the MinMax edge triangulation of all possible $\mathcal{A}$ is equal to the length of the maximum edge in $\mathcal{M}$.*

According to this lemma, in order to investigate the upper bound of edge length in $\mathcal{M}$, we only need to consider the maximum of maximal edge length in MinMax edge triangulation of $\mathcal{A}$. As $\mathcal{A}$ is for arbitrary boundary edge of $\boldsymbol{P}$, we turn to find the upper bound of the maximum edge in the MinMax edge triangulation of arbitrary $\mathcal{A}$.

Throughout this paper, we always use pq to denote the boundary edge in $\boldsymbol{P}$, and use p_1, q_1, respectively, to denote the lattice vertexes in $B(P')$ nearest to p and q. Sometimes we use the notation AB to directly denote the distance between point A and point B.

We begin with showing some properties of any polygon $\mathcal{A}$.

Lemma 2. *For any boundary edge pq of $\boldsymbol{P}$ in $\mathcal{A}$, there exists a vertex v on $B(P')$, such that either $0 \leq pv \leq l$ or $l < pv \leq \frac{2}{\sqrt{3}}l$. Furthermore, if pv satisfies $l < pv \leq \frac{2}{\sqrt{3}}l$ then the $\angle vpq$ in $\mathcal{A}$ is no more than $\frac{\pi}{2}$.*

Lemma 3. *Let $\mathcal{A}$ denote the polygon corresponding to the boundary edge L, and L_B be the lattice edge path on $\mathcal{A}$, and L_B^* be the length of the line segment connecting the two endpoints of L_B, then we have*

$$L_B^* \geq \frac{\sqrt{3}}{2} l \cdot n_L$$

where n_L denotes the number of lattice edges on L_B.

Lemma 4. *The number of edges in any polygon $\mathcal{A}$ is at most six.*

The following is a main theorem of this paper.

Theorem 1. *The maximum edge length in $\mathcal{M}$ is no more than $\frac{\sqrt{219}}{10}l$, and this upper bound is tight.*

Proof. We first summarize the proof. By Lemma 1, we may only need to investigate the upper bound of the maximum edge length in MinMax edge triangulation of $\mathcal{A}$. To this end, we show that for any case of $\mathcal{A}$, there exist a triangulation to make the length of maximum edge no more than $\frac{\sqrt{219}}{10}l$. Next for proving the tight upper bound, an actual $\mathcal{A}$ and its MinMax edge triangulation will be presented, whose maximum edge length in the triangulation is exactly $\frac{\sqrt{219}}{10}l$.

We now proceed with the details. If $p_1 = q_1$, that is, $\mathcal{A}$ is a triangle, the upper bound is αl. In the following we only consider the case that the number of edges in $\mathcal{A}$ is more than 3.

Recalling Lemma 4, $\mathcal{A}$ has at most six edges. The graph of $\mathcal{A}$ and its triangulation are just shown in Fig. 1, where $p_1A_1 = A_1A_2 = A_2q_1 = l$, and at the degenerate case, point p_1 may be equal to A_1, point q_1 coincides with A_2 and point A_1 may be equal to A_2. In the following we may only consider the non-degenerate cases since the degenerate one is a special case of non-degenerate cases. We draw the lines pA_1, pA_2 and qA_2 if $pA_2 \leq qA_1$ (see the left case of Fig. 1), or connect the line pA_1, qA_1 and qA_2 if $qA_1 < pA_2$ (see the right case of Fig. 1), to obtain the triangulation of $\mathcal{A}$. Without loss of generality, we assume $pA_2 \leq qA_1$ and only consider the left case of Fig. 1.

Firstly we have $pp_1 \leq pA_1$ and $qq_1 \leq qA_2$ by the definition of p_1 and q_1, so the possible maximal edge of triangulation is pq, pA_2, qA_2 or pA_1. We then distinguish the four different cases.

Case 1. The maximal edge is pq. For this case, the maximal edge length is αl and the upper bound is $1.4l$ as $\alpha \leq 1.4$.

Case 2. The maximal edge is pA_2. For this case, as $pA_2 \leq qA_1$, the length of pA_2 reaches its maximal length for the MinMax edge triangulation of $\mathcal{A}$, then the quadrilateral pqA_2A_1 is an isosceles trapezoid and the two edges pA_2 and

Fig. 1. Illustration used for the proof of Theorem 1: possible shapes of $\mathcal{A}$ and its triangulation. The left case is used for $pA_2 \leq qA_1$ and the right case is used for $qA_1 < pA_2$.

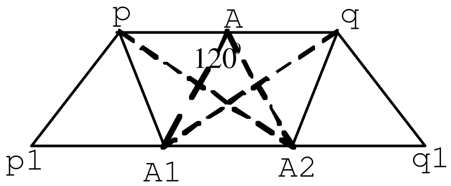

Fig. 2. Illustration used for the proof of Theorem 1, case 2.

qA_1 are the trapezoidal diagonals. In this case pq and A_1A_2 are parallel. So the length of pA_2 achieves the upper bound when the distance between pq and A_1A_2 reaches the maximum. The resulting $\mathcal{A}$ and its triangulation is shown in Fig. 2. According to cosine theorem in $\triangle ApA_2$, the upper bound of $pA_2$is

$$\left[(\frac{7}{10}l)^2 + l^2 - 2 \cdot \frac{7}{10}l \cdot l \cdot \cos(\frac{2}{3}\pi)\right]^{\frac{1}{2}} = \frac{\sqrt{219}}{10}l$$

Case 3. The maximal edge is qA_2. For this case, the upper bound is also $\frac{\sqrt{219}}{10}l$. The proof is done in the same manner as those given in case 2.

Case 4. The maximal edge is pA_1. For this case, we have $pA_1 \geq pq_1$ and $pA_1 \geq pA_2$ since pA_1 is the maximal edge. In the following we analyze the position of point "p" to show that this case does not happen.

Since $pA_1 \geq pq_1$, vertex p should belong to the left section of the midperpendicular line of p_1A_1. But vertex p also belongs to the right section of the midperpedicular line of A_1A_2 by $pA_1 \geq pA_2$. So vertex p must belong to the joint set of these two sections, that is, the polygon $\mathcal{A}$ must be like Fig. 3. However, in Fig. 3, vertex A is the nearest point to p, which contradicts the assumption that point p_1 is the point nearest to p. So pA_1 cannot be the maximal edge in $\mathcal{A}$.

Hence we have proved that the upper bound of maximum edge in MinMax edge triangulation of $\mathcal{A}$ is $\frac{\sqrt{219}}{10}l$, and from the Case 2 of proof, the tightness is obvious. □

By Theorem 1, we have obtained that the maximum edge length in triangulation $\mathcal{M}$ is no more than $\frac{\sqrt{219}}{10}l$. However, the lower bound of the edge length

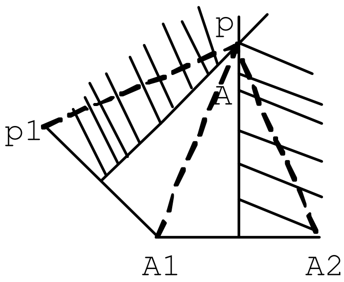

Fig. 3. Illustration used for the proof of Theorem 1, case 4.

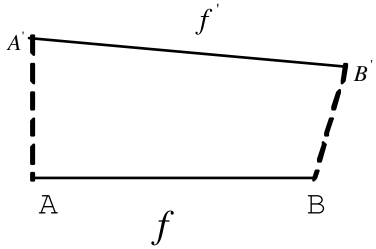

Fig. 4. Illustration used for the proof of Theorem 2, Case 3c.

has not be guaranteed in the obtained triangulation, i.e., some edges length in $\mathcal{M}$ may be very small. In the following we will consider the method to guarantee each edge length is no less than βl, where β is a given constant with $0 < \beta < \frac{1}{2}$.

3 A Triangulation with Edge Length No Less Than βl

We are now ready to show how triangulation $\mathcal{M}$ obtained by Heuristic A can be modified to give a solution for problem posed in the introduction. Theorem 1 implies the maximum edge length in $\mathcal{M}$ is bounded from above. Thus we only need to consider how to guarantee that edge lengths are bounded from below by βl. The key idea behind our heuristic is to simply contract those edges. (Note that we sometimes abuse f to denote the length of edge f.)

Heuristic B
Step 1-3: The same as Heuristic A. Denote the obtained triangulation by $\mathcal{M}$.
Step 4: For each edge f in $\mathcal{M}$, if $f < \beta l$ then one endpoint of f must be in P and the other must be in $B(P')$. Denote the endpoint of f in P by p and the endpoint in $B(P')$ by v, move v to p.

Note that for guaranteeing the existence of triangulation, we must let the "move" in Step 4 be "clockwise move" by the order of vertices of P. Let $\mathcal{N}$ denote the triangulation obtained by Heuristic B. The following theorem presents the length bound of edges in $\mathcal{N}$.

Theorem 2. *The edge lengths in triangulation $\mathcal{N}$ are in the interval*

$$\left[\beta l, \max\{l + 2\beta l, \frac{\sqrt{219}}{10} l + \beta l\}\right].$$

Proof. Since the lower bound βl is trivial, we need only to prove the upper bound. For each edge f in triangulation $\mathcal{M}$ of the polygon region between $CH(P)$ and $B(P')$, three cases are distinguished, according to the position of endpoints of f.

Case 1. Both of the two endpoints of f belong to P. For this case, edge f is an edge of $CH(P)$ and does not change by Heuristic B as $f \geq l$, thus $f \leq \alpha l \leq 1.4l$.

Case 2. One endpoint of f belongs to $\boldsymbol{P}$ and another endpoint of f belongs to $B(P')$.

Case2a: If the edge f do not change in $\mathcal{N}$, then we have $f \leq \frac{\sqrt{219}}{10}l$ by Theorem 1.

Case2b: Now assume the endpoint of edge f in $B(P')$ is moved, as the endpoint of f in $B(P')$ move to an vertex of $\boldsymbol{P}$, then the length of newly formed edges are bounded by $\frac{\sqrt{219}}{10}l + \beta l$ according to Theorem 1 and triangle inequality.

Case 3. Both of the two endpoints of f belong to $B(P')$.

Case3a: If edge f does not change in $\mathcal{N}$, then we have $f = l$.

Case3b: If only one endpoint of f changes in $\mathcal{N}$, the newly formed edges in $\mathcal{N}$ are no more than $\beta l + l$ according to triangle inequality.

Case3c: If both of the two endpoints of f moves in $\mathcal{N}$. See Fig. 4. Let edge f be AB, and let us assume vertex A moves to vertex A', vertex B moves to vertex B' and the newly formed edge f' is denoted by $A'B'$. The edges f, f', $A'A$ and $B'B$ formes a quadrangle. We have $AA' < \beta l$, $BB' < \beta l$ and $f = l$, thus triangle inequality gives $f' < A'A + AB + BB' < l + 2\beta l$.

Thus, the edge lengths of $\mathcal{N}$ are upper bounded by $\max\{l + 2\beta l, \frac{\sqrt{219}}{10}l + \beta l, \beta l + l, 1.4l, l\} = \max\{l + 2\beta l, \frac{\sqrt{219}}{10}l + \beta l\}$ and the theorem is proved. □

By Theorem 2 and $l + 2\beta l \leq 2l$, $\frac{\sqrt{219}}{10} + \beta l < 2l$, the Heuristic B is actually capable of generating the triangulation with all edges bounded by $[\beta l, 2l]$, thus meet the need of the primal problem.

4 On the Number of Non-standard Bars

To estimate the performance of $\mathcal{N}$, we consider the final procedure shown in Heuristic B. Since the number of edges in $\mathcal{N}$ is no more than the number of edges in $\mathcal{M}$, the number of non-standard bars is bounded by the number of edges in the triangulation of the region between $\boldsymbol{P}$ and $B(P')$.

Lemma 5. *The number of lattice edges on $B(P')$ is bounded by $\left\lceil \frac{2}{\sqrt{3}}\alpha \cdot n \right\rceil$.*

Lemma 6. *The number of edges on $CH(B(P'))$ is bounded by $\left\lceil \frac{2}{\sqrt{3}}\alpha \cdot n \right\rceil$.*

Theorem 3. *The number of edges in a triangulation of the region between $\boldsymbol{P}$ and $B(P')$ is bounded by $n + \left\lceil \frac{2}{\sqrt{3}}\alpha \cdot n \right\rceil$.*

Proof. Let S_1 denote the point set of P and S_2 denote the point set of P'. The Eulerian relation [7] for planar graph implies the following equalities:

$$|T(S_1 \cup S_2)| = 3|S_1 \cup S_2| - |CH(S_1 \cup S_2)| - 3$$
$$|T(S_2)| = 3|S_2| - |CH(S_2)| - 3$$

where $|T(S_1 \cup S_2)|$ and $|T(S_2)|$ denote the number of edges in triangulation $T(S_1 \cup S_2)$ and triangulation $T(S_2)$, respectively, $|S_1 \cup S_2|$ and $|S_2|$ denote the number of points in $S_1 \cup S_2$ and S_2, respectively, and $CH(S_1 \cup S_2)$ and $CH(S_2)$ are the number of edges in convex hull of $S_1 \cup S_2$ and S_2, respectively.

We have

$$|S_1 \cup S_2| = |S_1| + |S_2|,$$
$$|CH(S_1 \cup S_2)| = |P| = n,$$
$$|CH(S_2)| = |CH(B(P')| \leq \left\lceil \frac{2}{\sqrt{3}}\alpha \cdot n \right\rceil .$$

where the first equality uses $S_1 \cap S_2 = \emptyset$ and the final inequality uses Lemma 6. Then

$$\begin{aligned} |T(S_1 \cup S_2)| - |T(S_2)| &= 3|S_1 \cup S_2| - |CH(S_1 \cup S_2)| - 3|S_2| + |CH(S_2)| \\ &= 3|S_1| - n + |CH(S_2)| \\ &= 2n + |CH(S_2)| \\ &\leq 2n + \left\lceil \frac{2}{\sqrt{3}}\alpha n \right\rceil . \end{aligned}$$

where the third step uses the fact that the number of points in set S_1 is equal to n.

Thus we finish the proof by investigating that the number of edges in triangulation of the region between $\boldsymbol{P}$ and $B(P')$ is just $|T(S_1 \cup S_2)| - |T(S_2)|$ minus the number of edges of $\boldsymbol{P}$. □

Remark 1. If $B(P')$ is a convex polygon, then the number of lattice edges on $B(P')$ is bounded by $\lceil \alpha n \rceil$, and the number of edges in a triangulation of the region between P and $B(P')$ is bounded by $n + \lceil \alpha n \rceil$.

5 Conclusion and Future Work

In this paper, we have presented heuristics to generate a triangular mesh with the number of standard bars as many as possible. An interesting open problem is to investigate whether we can refine this procedure to obtain better results. What is more, our problem is a simple form of the following general problem:

For given real numbers $\alpha \leq \beta \leq \gamma$, and a convex polygon $\boldsymbol{P}$, how can we find a Steiner triangulation, $T(P)$, of P such that the length of inner edge in $T(P)$ is in the interval $[\alpha, \gamma]$ and the number of edges with edge length different from β is minimum?

All results given in this paper hold for polygon with boundary edge bounded by $[l, \alpha l]$ for $1 \leq \alpha \leq 1.4$, what is the largest value for α to let our results hold is still an open problem.

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References

1. F. Aurenhammer, N. Katoh, H. Kojima, M. Ohsaki and Y. F. Xu, Approximating uniform triangular meshes in polygons, *Theoretical Computer Science*, 289(2002), pp 879-895
2. M. Bern, D. Dobkin, D. Eppstein, Triangulating polygons without large angles, *Internat. J. Comput. Geom. Appl.* 5(1995) 171-192.
3. M. Bern, D. Eppstein, Mesh generation and optimal triangulation, in: D. -Z. Du (Ed.), *Computing in Euclidean Geometry*, World Scientific, Singapore, 1992, 47-123.
4. M. Bern, D. Eppstein, J. R. Gilbert, Provably good mesh generation, *J. Comput. System Sci.* 48 (1994) 384-409.
5. M. Bern, S. Mitchell, J. Ruppert, Linear-size nonobtuse triangulation of polygons, Proc. 10th Ann. ACM Symp. on Computational Geometry, 1994, 221-230.
6. P. Chew, Guaranteed-quality mesh generation for curved surfaces, Proc. 9th Ann. ACM Symp. on Computational Geometry, 1993, 274-280.
7. H. Edelsbrunner, *Algorithms in Combinatiorial Geometry*, EATCS Monographs on Theoretical Cmputer Science 10, Springer-Verlag, 1987
8. P. D. Gilbert, *New results in planar triangulation*, report R-850, Coordinated Science Laboratory, University of Illinois, 1979.
9. G. T. Klincsek, Minimal triangulations of polygonal domains, *Ann. Discrete Math*, 9, 127-128, 1980
10. E. Melisseratos, D. Souvaine, Coping with inconsistencies: a new approach to produce quality triangulations of polygonal domains with holes, Proc. 8th Ann. ACM Symp. on Computational Geometry, 1992, 202-211.
11. J. Ruppert, A Delaunay refinement algorithm for quality 2-dimensional mesh generation, *J. Algorithms* 18 (1995) 548-585.