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On Graphs with Zero Determinant of Adjacency Matrices*

Xu Yinfeng(徐寅峰)
(*The School of Management, Xi'an Jiaotong University, Xi'an*, 710049)
Dong Fengming(董峰明)
(*Institute of Applied Mathematics, Academia Sinica, Peijing*, 100080)

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In this paper, we only consider simple graphs. For a given graph G, let $A(G)$ be its adjacency matrix. Though a lot of algebraic properties of graphs have been discussed in some papers in [1,2], the sufficient and necessary conditions for graphs with zero determinant of adjacent matrices are not concerned. In this paper, we consider graphs with at most one circuit. The sufficient and necessary conditions are found for such graphs to have zero determinant of adjacent matrices.

Definition 1 The even graph with the smallest rank is K_2. Let G be a graph with $G=n \geqslant 4$, G is an even graph on n vertices if and only if there exists a pendant vertex x_0 on G such that $G-x_0-y_0$ is an even graph on $n-2$ vertices, where y_0 is the vertex adjacent to x_0.

From this definition, the following corollaries can be found:

Corollary 1 A graph is an even graph if and only if all of its components are even graphs.

Corollary 2 G is an even graph iff for any pendent vertex x, $G-x-x_0$ is an even graph, where x_0 is the vertex adjacent to x.

Proof By definition 1, the sufficient condition is obvious. Now we prove the necessary condition. When $|G|\leqslant 4$. we can check this result. Let G be an even graph with at least 6 vertices, x be a pendent vertex on G and x_0 be the vertex adjacent to x. By Definition 1, there exists a pendent vertex y on G such that $G-y-y_0$ is an even graph, where y_0 is the vertex adjacent to y. Here, we suppose $x\neq y$, then $y_0\neq x_0$. Otherwise, by corollary 1, G is not an even graph. By inductive hypothesis, $G-x-x_0-y-y_0$ is an even graph. Therefore, $G-x-x_0$ is an

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even graph by Definition 1.

Theorem 1 Let G be a graph with a pendent vertex x, and $G_1, \cdots, G_m$ be the all components of $G-x-x_0$, where x_0 is the vertex adjacent to x. Then

$$\det(A(G)) = -\prod_{i=1}^{m} \det(A(G_i)).$$

This theorem can be proved easily from the theory of linear algebra.

From Theorem 1, we get

Corollary 3 If G is an even graph, then $|\det(A(G))| = 1$.

Theorem 2 Let T be a tree. Then

$$|\det(A(T))| = \begin{cases} 1, \text{if } T \text{ is an even graph}, \\ 0, \text{otherwise}. \end{cases}$$

Proof If $|T| \leqslant 5$, we can check this result easily. Now, let $|T| \geqslant 6$, x be a pendent vertex on T and x_0 be the vertex adjacent to x. By Theorem 1 and inductive hypothesis, $\det(A(T - x - x_0)) = 0$ if there exists a component T_0 on $T - x - x_0$ such that T_0 is not an even graph. Otherwise, $|\det(A(T - x - x_0))| = 1$. By definition 1 and theorem 1, this theorem is proved.

Lemma 1[2] Let C_n be a circuit graph on n vertices, then $\det(A(C_n)) = 0$ iff $n \equiv 0 \pmod 4$.

In the following, we will prove the main result in this paper.

Theorem 3 Let G be a connected graph with only one circuit C, then $\det(A(G)) \neq 0$ iff

(1) $|C| \not\equiv 0 \pmod 4$ and $G-C$ is an even graph or $G=C$, or

(2) G is an even graph.

Proof If $G=C$ or each component of $G-C$ is a vertex, then the theorem is proved easily. Now we suppose one of the components of $G-C$ has more than one vertex. Let G_0 be one such component, $x \in V(G_0)$ be a pendent vertex of G and y be the vertex adjacent to x. It is obvious that y is not on C. One of the components of $G-x-y$ contains C and the others are trees. If one of these trees is not an even graph, then neither $G-C$ nor G are even graphs and $\det(A(G)) = 0$ by Theorem 2. So $\det(A(G)) \neq 0$ iff all the components $G_1, G_2, \cdots, G_k$ of $G-x-y$ which don't contain C are even graphs and $\det(A(G - x - y - G_1 - G_2 - \cdots - G_k)) \neq 0$. By inductive hypothesis, Corollary 1, and Theorem 2, the theorem is proved.

References

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