A Risk-Reward Competitive Analysis for the Recoverable Canadian Traveller Problem*

Bing Su[1,2], Yinfeng Xu[1,3], Peng Xiao[1,3], and Lei Tian[1]

[1] School of Management, Xi'an Jiaotong University, Xi'an, 710049, P.R. China
[2] School of Economics and Management, Xi'an Technological University, Xi'an, 710032, P.R. China
[3] The State Key Lab for Manufacturing Systems Engineering, Xi'an, 710049, P.R. China
{subing,yfxu,xiaopeng}@mail.xjtu.edu.cn, ttianlei@163.com

Abstract. From the online point of view, we study the Recoverable Canadian Traveller Problem (Recoverable-CTP) in a special network, in which the traveller knows in advance the structure of the network and the travel time of each edge. However, some edges may be blocked and the traveller only observes that upon reaching the vertex of the blocked edge, and the blocked edge may be reopened but the traveller doesn't know its recovery time. The goal is to find a least-cost route from the origin node to the destination node, more precisely, to find an adaptive strategy minimizing the ratio of traversed time to the travel time of the optimal offline shortest path (where all blocked edges and their recovery time are known in advance). We present an optimal online strategy - a comparison strategy and prove its competitive ratio. Moreover, with the different forecasts of the recovery time, some online strategies are given under the risk-reward framework, and the rewards and the risks of the different strategies are analysed.

Keywords: Recoverable-CTP, Competitive analysis, Comparison strategy, Risk-reward model.

1 Introduction

The Canadian Traveller Problem (CTP) has been introduced in [1] and is defined as follows: Suppose that a traveller knows in advance the structure of a network and the travel time of each edge. However, some edges may fail and the traveller only observes that upon reaching a vertex of the blocked/failed edge. The problem is to devise a good travel strategy from the origin node to the destination node without any knowledge of future edge blockages. Under this setting, Papadimitriou and Yannakakis proved that devising an online algorithm with a bounded competitive ratio is PSPACE-complete [1].

* The authors would like to acknowledge the support of research grant No. 70525004, 60736027, 70121001 from the NSF, No. 20060401003 from the PSF Of China and No. 06JK099 from the Education Department of Shaanxi.

B. Yang, D.-Z. Du, and C.A. Wang (Eds.): COCOA 2008, LNCS 5165, pp. 417–426, 2008.

Several variations of the CTP were studied in [2-5]. If there is a given parameter k which bounds the number of blocked edges from above, the resulting problem is called the k-Canadian Traveller Problem (k-CTP) [2]. Bar-Noy and Schieber studied the k-CTP, but they did not consider the problem from a competitive analysis point of view. Instead, they considered the worst-case criterion which aims at a strategy where the maximum cost was minimized [2]. Westphal considered the online version of k-CTP and showed that no deterministic online algorithm can achieve a competitive ratio smaller than $2k + 1$ and gave an easy algorithm which matches this lower bound [3]. The same bound was in fact obtained independently in [4]. Westphal also showed that randomization can not improve the competitive ratio substantially. He showed that by establishing a lower bound of for the competitiveness of randomized online algorithms against an oblivious adversary [3]. Recoverable-CTP is a variation of CTP, in which the blocked edges may be reopened [2]. Under the assumptions that the upper bound on the number of blockages is known in advance and the recovery time are not very long compared with the travel time, Bar-Noy and Schieber presented a polynomial-time travel strategy which guarantees the shortest worst-case travel time. For the Stochastic Recoverable CTP, again when the recovery time are not very long relative to the travel time, they also presented a polynomial-time strategy which minimizes the expected travel time [2]. The online strategies were studied for the Recoverable-CTP in general networks in [5]. Under the assumption that the traveller doesn't know the recovery time upon reaching a vertex of the blocked edge, two adaptive strategies - a waiting strategy and a greedy strategy were presented, and the competitive ratios for the two strategies were given, respectively [5].

In this paper, we focus on the online version of the Recoverable-CTP in a special network, in which the vertex set is $V = \{v_1, v_2, \cdots, v_n\}$ and there are multiple edges between v_i and v_{i+1}, where $i = 1, \cdots, n - 1$. Some edges may be blocked and the traveller only observes that upon reaching the vertex of the blocked edge, and the blocked edge may be reopened but the traveller doesn't know its recovery time. Our goal is to find a least-cost route from the origin node to the destination node by passing through $v_1, v_2, \cdots, v_n$ one by one, more precisely, to find an adaptive strategy minimizing the competitive ratio, which compares the performance of this strategy with that of a hypothetical offline algorithm that knows the entire topology in advance. We present an optimal online strategy - a comparison strategy and prove its competitive ratio for the Recoverable-CTP. Moreover, with the different forecasts of the recovery time, some online strategies are given under the risk-reward framework, and the rewards and the risks of different strategies are analysed.

The organization of this paper is as follows. In Section 2, the problem definition and some assumptions are briefly reviewed. In Section 3, we propose and investigate optimal online strategies for the Recoverable-CTP in a special network. In Section 4, we consider the performance of some strategies under the risk-reward framework. Finally, we conclude the work in Section 5.

2 Problem Statement and Formulation

Let G be an undirected network with $|V| = n$ vertexes, where $V = \{v_1, v_2, \cdots, v_n\}$. Let v_1 be the origin and v_n the destination. Let $e_{i,m^i} = \{e_{i,1}, \cdots, e_{i,j}, \cdots, e_{i,m^i}\}$ denote the set of all edges between v_i and v_{i+1}, where $e_{i,j}$ is the j shortest edge from v_i to v_{i+1}. Let $t_{i,j}$ denote the travel time of $e_{i,j}$. Therefore, $t_{i,1} \leq \cdots \leq t_{i,j} \cdots \leq t_{i,m^j}$. Let $P_1 = \{v_1, e_{1,1}, \cdots, v_i, e_{i,1}, \cdots, v_{n-1}, e_{n-1,1}, v_n\}$ be the shortest path from v_1 to v_n. Denote $\delta = (\delta_1, \delta_2, \cdots, \delta_k)$ as the blockages sequence, and $t_{k,r}$ as the recovery time with respect to δ_k. As shown in Fig. 1, suppose that blockages happen at P_1 and the traveller has to move from v_1 to v_n by passing through $v_2, v_3, \cdots, v_{n-1}$ one by one, then the problem is to design a good travel strategy without any knowledge of future blockages.

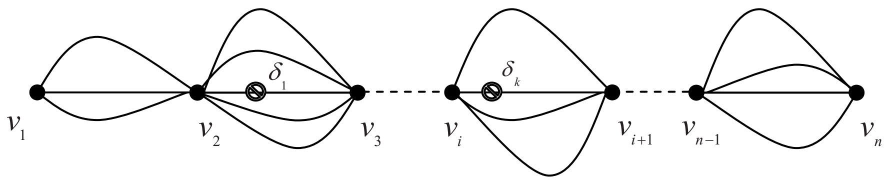

Fig. 1. Graph G

In order to discuss the problem, we make the following assumptions:

(1) The traveller knows the entire network and the travel time of each edge in advance.
(2) Blockages may happen at $P_1 = \{v_1, e_{1,1}, \cdots, v_i, e_{i,1}, \cdots, v_{n-1}, e_{n-1,1}, v_n\}$ and the traveller does not know which one edge will be blocked in advance, and the traveller only observes that upon arriving the vertex of the blocked edge.
(3) The recovery time of a blockage is not known in advance, but the traveller can obtain the recovery time until the blockage is reopened.
(4) Blockages may not happen at $e_{i,j}$ $(j \neq 1)$ between v_i and v_{i+1}.

If all of the blockages and their recovery time are known in advance, then the problem becomes an offline problem, and the optimal travel strategy is obtained by following the shortest edge from v_1 to v_n after modifying the travel time of known blocked edges. If the blocked edges are unpredictable, then the problem is obviously an online problem.

Let $C_{OPT}(\delta)$ be the travel time of the optimal offline shortest path from v_1 to v_n, and let $C_A(\delta)$ be the corresponding travel time of the online strategy A for the traveller to go from v_1 to v_n. Strategy A is said to be α-competitive [6-9] if $C_A(\delta) \leq \alpha_A \cdot C_{OPT}(\delta) + b$ holds, where α_A and b are constants not related to δ. Denote $\alpha^\star$ as the optimal competitive ratio for the on-line problem such that $\alpha^\star = \inf_{A \in S}(\alpha_A)$ for any online strategy $A \in S$, where S is the set of all online strategies. If $\alpha_{A^\star} = \alpha^\star$, then $A^\star$ is called the optimal online algorithm.

The above competitive analysis is the most fundamental and significant approach. However, the above competitive analysis is not very flexible, especially in the uncertainty environment. In practice, many travellers hope to manage their risk and willing to take certain risks for more rewards sometimes. Al-Binali [10] first defined the concepts of risk and reward for online financial problems. From the risk-reward point of view, The following definitions are given for our problem. Let I be the range of the recovery time of a blockage and $F \subset I$ be a forecast for the recovery time. If $F \subset I$ is the correct forecast, then denote $\hat{\alpha}_{\hat{A}} = \sup_{t_{k,r} \subset F} \frac{C_{\hat{A}}(\delta)}{C_{OPT}(\delta)}$ as the restricted competitive ratio of $\hat{A}$, and $f_{\hat{A}} = \frac{\alpha^\star}{\hat{\alpha}_{\hat{A}}}$ the reward of $\hat{A}$. The optimal restricted competitive ratio under the forecast F is $\hat{\alpha}^\star = \inf_{\hat{A} \in S} (\hat{\alpha}_{\hat{A}})$. If $F \subset I$ is the false forecast, then denote $\alpha_{\hat{A}}$ be the competitive ratio of $\hat{A}$ for any $t_{k,r} \in I$. Define $\tau_{\hat{A}} = \frac{\alpha_{\hat{A}}}{\alpha^\star}$ as the risk of $\hat{A}$.

3 Competitive Analysis of the Comparison Strategy

In this section, we present an optimal online strategy for the Recoverable-CTP in a special network and analyse its corresponding competitive ratio.

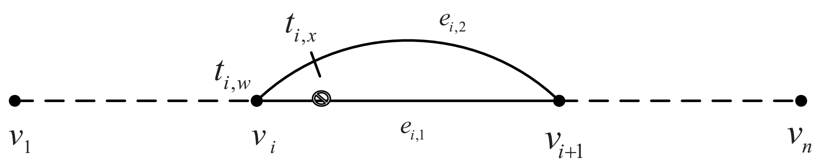

Fig. 2. Analysis of CS

Comparison Strategy: When the traveller reaches v_i and knows that the edge $e_{i,1}$ is blocked, he/she sets a upper bound of waiting time $t_{i,w} = \frac{t_{i,2} - t_{i,1}}{2}$ at v_i. If the recovery time $t_{i,r}$ of $e_{i,1}$ satisfies $t_{i,r} \leq t_{i,w}$, then the traveller follows the shortest edge $e_{i,1}$ after waiting time $t_{i,r}$; otherwise, the traveller follows the second shortest edge $e_{i,2}$ and when he/she knows that the blockage is reopened after traversed time $t_{i,x}$ ($t_{i,x} < t_{i,2}$) in $e_{i,2}$, he/she makes a decision according to the following condition: If $t_{i,1} + t_{i,x} \geq t_{i,2} - t_{i,x}$, he/she continues on the second shortest edge $e_{i,2}$ as intended; If $t_{i,1} + t_{i,x} \leq t_{i,2} - t_{i,x}$, then he/she returns to v_i and follows the shortest edge $e_{i,1}$.

Denote the Comparison Strategy as CS. As shown in Fig.2.

According to the above travelling strategy CS, we have the following lemma and theorems.

Lemma 1. If $t_{i,w} \leq t_{i,r} \leq t_{i,2} - t_{i,1}$ and $t_{i,x} < \frac{t_{i,2} - t_{i,1}}{2}$, he/she returns to v_i and follows the shortest edge $e_{i,1}$ by using CS.

Proof. If $t_{i,r} \geq t_{i,w}$, then the traveller follows the second shortest edge $e_{i,2}$ from v_i to v_{i+1} by using CS. When he/she knows that the blockage is reopened, he/she returns to v_i and follows the shortest edge $e_{i,1}$ under the condition $t_{i,x} + t_{i,1} < t_{i,2} - t_{i,x}$. Since $t_{i,r} = t_{i,x} + \frac{t_{i,2}-t_{i,1}}{2}$, we have $t_{i,r} < t_{i,2} - t_{i,1}$. Hence, if $t_{i,w} \leq t_{i,r} \leq t_{i,2} - t_{i,1}$, then the traveller returns to v_i and follows the shortest edge $e_{i,1}$.

Theorem 1. For the Recoverable-CTP with the blockage sequence $\delta = (\delta_1, \delta_2, \cdots, \delta_k)$, the competitive ratio of CS is $\frac{3-\beta}{2}$, where $\beta = \max \beta_i$, $\beta_i = \frac{t_{i,1}}{t_{i,2}}$.

Proof. Set $t_{i,w} = \frac{t_{i,2}-t_{i,1}}{2}$.

(1) If $t_{i,r} < t_{i,w}$, then the online traveller waits $t_{i,r}$ time at v_i and follows the shortest edge $e_{i,1}$, and we have $C_{CS}(\delta_i) = t_{i,r} + t_{i,1}$. The offline optimal strategy is the same as the online strategy. Hence, we have $C_{OPT}(\delta_i) = t_{i,r} + t_{i,1}$ and $\frac{C_{CS}(\delta_i)}{C_{OPT}(\delta_i)} = 1$.

(2) If $t_{i,w} \leq t_{i,r} < t_{i,2} - t_{i,1}$, then the online traveller waits $t_{i,w}$ time at v_i and follows the second shortest edge $e_{i,2}$. When he/she knows that the blockage is reopened, he/she returns to v_i and follows the shortest edge $e_{i,1}$ by Lemma 1. The offline optimal strategy is that the traveller waits $t_{i,r}$ time at v_i and follows the shortest edge $e_{i,1}$. Hence, we have $C_{CS}(\delta_i) = t_{i,r} + t_{i,1} + t_{i,x}$, $C_{OPT}(\delta_i) = t_{i,r} + t_{i,1}$ and $C_{CS}(\delta_i) = (1 + \frac{t_{i,x}}{t_{i,r}+t_{i,1}})C_{OPT}(\delta_i)$. Since $t_{i,r} < t_{i,2} - t_{i,1}$ and $t_{i,x} < \frac{t_{i,2}-t_{i,1}}{2}$, we have $\frac{C_{CS}(\delta_i)}{C_{OPT}(\delta_i)} \leq \frac{3}{2} - \frac{t_{i,1}}{2t_{i,2}} = \frac{3-\beta_i}{2}$, where $\beta_i = \frac{t_{i,1}}{t_{i,2}}$.

(3) If $t_{i,r} \geq t_{i,2} - t_{i,1}$, then the online traveller waits $t_{i,w}$ time at v_i and follows the second shortest edge $e_{i,2}$. The offline optimal strategy is that the traveller follows the second shortest edge $e_{i,2}$ without any waiting time. Hence, we have $C_{CS}(\delta_i) = t_{i,w} + t_{i,1} = \frac{t_{i,2}+t_{i,1}}{2} \leq \frac{3t_{i,2}-t_{i,1}}{2}$, $C_{OPT}(\delta_i) = t_{i,2}$ and $\frac{C_{CS}(\delta_i)}{C_{OPT}(\delta_i)} \leq \frac{3}{2} - \frac{t_{i,1}}{2t_{i,2}} \leq \frac{3-\beta_i}{2}$, where $\beta_i = \frac{t_{i,1}}{t_{i,2}}$.

From (1), (2) and (3), we have $\frac{C_{CS}(\delta_i)}{C_{OPT}(\delta_i)} = \frac{\sum_{i=1}^{k} C_{CS}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} \leq \frac{3-\beta}{2}$, where $\beta = \max \beta_i$ and $\beta_i = \frac{t_{i,1}}{t_{i,2}}$. Therefore, the competitive ratio of CS is $\frac{3-\beta}{2}$.

This concludes the proof of Theorem 1. □

Theorem 2. For the Recoverable-CTP with the blockage sequence $\delta = (\delta_1, \delta_2, \cdots, \delta_k)$, the competitive ratio h of any deterministic online strategy satisfies $h \geq \frac{3-\beta}{2}$.

Proof. For any online deterministic strategy B, let $t_{i,B}$ be the upper bound of waiting time when the traveller reaches v_i of blocked edge $e_{i,1}$. Set $t_{i,w} = \frac{t_{i,2}-t_{i,1}}{2}$, and consider three cases.

(1) $t_{i,B} \in [0, t_{i,w}]$
The worst case is that the online traveller knows the blockage reopened after traversed time $\frac{t_{i,2}-t_{i,1}}{2}$ along $e_{i,2}$, he/she can continue on the second shortest edge $e_{i,2}$ or return to v_i and follow the shortest edge $e_{i,1}$. The travel

time is $C_B(\delta_i) = t_{i,B} + \frac{t_{i,2}-t_{i,1}}{2} + \frac{t_{i,2}-t_{i,1}}{2} + t_{i,1} = t_{i,B} + t_{i,2}$. The offline optimal strategy is that the traveller follows the shortest edge $e_{i,1}$ after waiting time $t_{i,r} = t_{i,B} + \frac{t_{i,2}-t_{i,1}}{2}$, and we have $C_{OPT}(\delta_i) = t_{i,B} + \frac{t_{i,2}-t_{i,1}}{2} + t_{i,1} = t_{i,B} + \frac{t_{i,2}+t_{i,1}}{2}$. Since $t_{i,B} < \frac{t_{i,2}-t_{i,1}}{2}$, we have $\alpha_{i,1} = \frac{C_B(\delta_i)}{C_{OPT}(\delta_i)} = \frac{t_{i,B}+t_{i,2}}{t_{i,B}+\frac{t_{i,2}+t_{i,1}}{2}} > \frac{\frac{t_{i,2}-t_{i,1}}{2}+t_{i,2}}{\frac{t_{i,2}-t_{i,1}}{2}+\frac{t_{i,2}+t_{i,1}}{2}} = \frac{3t_{i,2}-t_{i,1}}{2t_{i,2}} = \frac{3-\beta_i}{2}$, where $\beta_i = \frac{t_{i,1}}{t_{i,2}}$. Hence, we have $\alpha_{B,1} = \frac{\sum_{i=1}^{k} C_B(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} > \frac{3-\beta}{2}$, where $\beta = \max \beta_i$.

(2) $t_{i,B} = t_{i,w}$
If $t_{i,B} = t_{i,w}$, then by Theorem 1, we have the competitive ratio of B is $\alpha_{B,2} = \frac{3-\beta}{2}$.

(3) $t_{i,B} \in (t_{i,w}, +\infty)$
The worst case is that the online traveller follows the second shortest edge $e_{i,2}$ after waiting time $t_{i,B}$ by using online strategy B. The offline optimal strategy is that the traveller follows the second shortest edge $e_{i,2}$ without any waiting time. Hence, we have $C_B(\delta_i) = t_{i,B}+t_{i,2}$, $C_{OPT}(\delta_i) = t_{i,2}$ and $\alpha_{i,3} = \frac{C_B(\delta_i)}{C_{OPT}(\delta_i)} = 1 + \frac{t_{i,B}}{t_{i,2}}$. Since $t_{i,B} > \frac{t_{i,2}-t_{i,1}}{2}$, we have $\alpha_{i,3} > 1 + \frac{t_{i,2}-t_(i,1)}{2t_{i,2}} = \frac{3-\beta_i}{2}$, where $\beta_i = \frac{t_{i,1}}{t_{i,2}}$. Hence, we have $\alpha_{B,3} = \frac{\sum_{i=1}^{k} C_B(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} > \frac{3-\beta}{2}$, where $\beta = \max \beta_i$.

From the above analysis, we have $\alpha_B = \min\{\alpha_{B,1}, \alpha_{B,2}, \alpha_{B,3}\}$. Then the competitive ratio of any deterministic online strategy is no less than $\frac{3-\beta}{2}$.

This concludes the proof of Theorem 2. □

From the above analysis, it is known that the comparison strategy is the optimal deterministic online strategy, and $\alpha^\star = \frac{3-\beta}{2}$.

4 Competitive Analysis of the Risk-Reward Strategies

When an online traveller is risk-averse, he will use the classical online algorithm A and achieve the optimal competitive ratio. If the online traveller is a risk-seeker, then the risk-reward strategy allows him to benefit from his capability, and allows him to control his risk by using a risk strategy $\hat{A}$. Next, we will give the risk-reward strategy $\hat{A}$ with respect to the Recoverable-CTP and analysis it's competitive.

The online traveller can make three different forecasts of the recovery time upon reaching the blockage: $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$, $\frac{t_{i,2}-t_{i,1}}{2} \leq t_{i,r} < t_{i,2} - t_{i,1}$ and $t_{i,r} > t_{i,2} - t_{i,1}$, we will discuss the three cases as following.

Forecast 1. $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$.

For the forecast 1, we give the following online strategy $\hat{A}_1$.

$\hat{A}_1$**:** Set the upper bound of waiting time $\hat{t}_{i,w} = \frac{t_{i,2}-t_{i,1}}{2}$. If $t_{i,r} \leq \hat{t}_{i,w}$, then the traveller follows the shortest edge $e_{i,1}$ after waiting time $t_{i,r}$; otherwise, the traveller follows the second shortest edge $e_{i,2}$ and when he/she knows the blockage being reopened after traversed time $t_{i,x}$ ($t_{i,x} < t_{i,2}$) in $e_{i,2}$, he/she makes a decision according to the following condition: If $t_{i,1} + t_{i,x} \geq t_{i,2} - t_{i,x}$, he/she continues on the second shortest edge $e_{i,2}$ as intended; If $t_{i,1} + t_{i,x} < t_{i,2} - t_{i,x}$, then he/she returns to v_i and follows the shortest edge $e_{i,1}$.

Theorem 3. For the Recoverable-CTP with a correct forecast of the recovery time $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$, the restricted optimal competitive ratio of $\hat{A}_1$ is 1.

Proof. Set the upper bound of waiting time $\hat{t}_{i,2} = \frac{t_{i,2}-t_{i,1}}{2}$. If the forecast is correct, then $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$. The online traveller waits $t_{i,r}$ time at v_i and follows the shortest edge $e_{i,1}$, and we have $C_{\hat{A}_1}(\delta_i) = t_{i,r} + t_{i,1}$. The offline optimal strategy is the same as the online strategy, and we have $C_{OPT}(\delta_i) = t_{i,r} + t_{i,1}$. Hence, the restricted competitive ratio for the Recoverable-CTP is $\hat{\alpha}_{\hat{A}_1} = \frac{\sum_{i=1}^{k} C_{\hat{A}_1}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} = 1.$

This concludes the proof of Theorem 3. □

From the above theorem, It is known that the reward of the strategy $\hat{A}_1$ is $f_{\hat{A}_1} = \frac{\alpha^\star}{\hat{\alpha}_{\hat{A}_1}} = \frac{3-\beta}{2}$.

If the traveller makes a false forecast regarding the recovery time of the blockage, then $t_{i,r} > \frac{t_{i,2}-t_{i,1}}{2}$, and the competitive ratio of $\hat{A}_1$ is $\alpha_{\hat{A}_1} = \frac{3-\beta}{2}$ by Theorem 1. Hence, $\tau = \frac{\alpha_{\hat{A}_1}}{\alpha^\star} = 1$.

Forecast 2. $\frac{t_{i,2}-t_{i,1}}{2} \leq t_{i,r} < t_{i,2} - t_{i,1}$

For the forecast 2, we give the following online strategy $\hat{A}_2$.

$\hat{A}_2$**:** Set the upper bound of waiting time $\hat{t}_{i,w} = t_{i,2} - t_{i,1}$. If $t_{i,r} \leq \hat{t}_{i,w}$, then the traveller follows the shortest edge $e_{i,1}$ after waiting time $t_{i,r}$; otherwise, the traveller follows the second shortest edge $e_{i,2}$.

Theorem 4. For the Recoverable-CTP with a correct forecast of the recovery time $\frac{t_{i,2}-t_{i,1}}{2} \leq t_{i,r} < t_{i,2} - t_{i,1}$, the restricted optimal competitive ratio of $\hat{A}_2$ is 1.

Proof. Set the upper bound of waiting time $\hat{t}_{i,w} = t_{i,2} - t_{i,1}$. If the forecast is correct, then $\frac{t_{i,2}-t_{i,1}}{2} \leq t_{i,r} < t_{i,2} - t_{i,1}$. The online traveller waits $t_{i,r}$ time at v_i and follows the shortest edge $e_{i,1}$, and we have $C_{\hat{A}_2}(\delta_i) = t_{i,r} + t_{i,1}$. The offline optimal strategy is the same as the online strategy, and we have $C_{OPT}(\delta_i) = t_{i,r} + t_{i,1}$. Hence, the restricted competitive ratio for the Recoverable-CTP is $\hat{\alpha}_{\hat{A}_2} = \frac{\sum_{i=1}^{k} C_{\hat{A}_2}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} = 1.$

This concludes the proof of Theorem 4. □

From the above proof, we can obtain the reward of the strategy $\hat{A}_2$ is $f_{\hat{A}_2} = \frac{\alpha^\star}{\hat{\alpha}_{\hat{A}_2}} = \frac{3-\beta}{2}$.

If the traveller makes a false forecast regarding the recovery time of the blockage, then $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$ or $t_{i,r} \geq t_{i,2} - t_{i,1}$. If $t_{i,r} < \frac{t_{i,2}-t_{i,1}}{2}$, then the online traveller follows the shortest edge $e_{i,1}$ after waiting time $t_{i,r}$ by using online strategy $\hat{A}_2$, the offline optimal strategy is the same as the online strategy. Hence, the competitive ratio is 1, and $\tau_1 = \frac{\alpha_{\hat{A}_2}}{\alpha^\star} = \frac{1}{\frac{3-\beta}{2}} = \frac{2}{3-\beta}$. If $t_{i,r} \geq t_{i,2} - t_{i,1}$, then the online traveller follows the second shortest edge $e_{i,2}$ after waiting time $\hat{t}_{i,w}$ by using online strategy $\hat{A}_2$, the offline optimal strategy is that the traveller follows the second shortest edge without any waiting time. The competitive ratio of $\hat{A}_2$ is $\alpha_{\hat{A}_2} = \frac{\sum_{i=1}^{k} C_{\hat{A}_2}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} = \frac{\sum_{i=1}^{k}(t_{i,2}-t_{i,1}+t_{i,2})}{\sum_{i=1}^{k} t_{i,2}} \leq 2-\beta$, and $\tau_2 = \frac{\alpha_{\hat{A}_2}}{\alpha^\star} = \frac{2-\beta}{\frac{3-\beta}{2}} = \frac{4-2\beta}{3-\beta}$. Since $\tau_1 \leq \tau_2$, we have $\tau = \max\{\tau_1, \tau_2\} = \frac{4-2\beta}{3-\beta}$.

Forecast 3. $t_{i,r} > t_{i,2} - t_{i,1}$

For the forecast 3, we give the following online strategy $\hat{A}_3$.

$\hat{A}_3$**:** Set the upper bound of waiting time $\hat{t}_{i,w} = 0$. The traveller follows the second shortest edge $e_{i,2}$.

Theorem 5. For the Recoverable-CTP with a correct forecast of the recovery time $t_{i,r} > t_{i,2} - t_{i,1}$, the restricted optimal competitive ratio of $\hat{A}_3$ is 1.

Proof. Set the upper bound of waiting time $\hat{t}_{i,w} = 0$. If the forecast is correct, then $t_{i,r} > t_{i,2} - t_{i,1}$. The online traveller follows the second shortest edge $e_{i,2}$, and we have $C_{\hat{A}_3}(\delta_i) = t_{i,2}$. The offline optimal strategy is the same as the online strategy, and we have $C_{OPT}(\delta_i) = t_{i,2}$. Hence, the restricted optimal competitive ratio for the Recoverable-CTP is $\hat{\alpha}_{\hat{A}_3} = \frac{\sum_{i=1}^{k} C_{\hat{A}_3}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} = 1$.

This conclues the proof of Theorem 5. □

From the above proof, we can obtain the reward of the strategy $\hat{A}$ is $f_{\hat{A}_3} = \frac{\alpha^\star}{\hat{\alpha}_{\hat{A}_3}} = \frac{3-\beta}{2}$.

If the traveller makes a false forecast regarding the recovery time of the blockage, then $t_{i,r} < t_{i,2} - t_{i,1}$. The traveller follows the second shortest edge $e_{i,2}$ by using strategy $\hat{A}_3$. The worst case is that he/she knows the blockage reopened after traversed time $\frac{t_{i,2}-t_{i,1}}{2}$, he/she can continue on the second shortest edge $e_{i,2}$ or return to v_i and follow the shortest edge $e_{i,1}$. The total travel time is $C_{\hat{A}_3}(\delta_i) = t_{i,2}$. The offline optimal strategy is that the traveller follows the shortest edge $e_{i,1}$ after waiting time $\frac{t_{i,2}-t_{i,1}}{2}$, and we have $C_{OPT}(\delta_i) = \frac{t_{i,2}-t_{i,1}}{2} + t_{i,1} = \frac{t_{i,2}+t_{i,1}}{2}$.

The competitive ratio of $\hat{A}_3$ is $\alpha_{\hat{A}_3} = \frac{\sum_{i=1}^{k} C_{\hat{A}_3}(\delta_i)}{\sum_{i=1}^{k} C_{OPT}(\delta_i)} = \frac{\sum_{i=1}^{k} t_{i,2}}{\sum_{i=1}^{k} \frac{t_{i,2}+t_{i,1}}{2}} \leq \frac{2}{1+\beta}$, and $\tau = \frac{\alpha_{\hat{A}_3}}{\alpha^{\star}} = \frac{\frac{2}{1+\beta}}{\frac{3-\beta}{2}} = \frac{4}{(3-\beta)(1+\beta)}$.

From the above analysis, we conclude the results as shown in Table 1.

Table 1. Risk-reward strategy and its competitive analysis

Forecast	Strategy $\hat{A}$	$\hat{\alpha}_{\hat{A}} = \hat{\alpha}^*$	$f_{\hat{A}} = \frac{\alpha^*}{\hat{\alpha}_{\hat{A}}}$	$\alpha_{\hat{A}}$	$\tau = \frac{\alpha_{\hat{A}}}{\alpha^*}$
$t_{i,r} \leq \frac{t_{i,2}-t_{i,1}}{2}$	$\hat{t}_{i,w} = \frac{t_{i,2}-t_{i,1}}{2}$	1	$\frac{3-\beta}{2}$	$\frac{3-\beta}{2}$	1
$\frac{t_{i,2}-t_{i,1}}{2} < t_{i,r} \leq t_{i,2}-t_{i,1}$	$\hat{t}_{i,w} = t_{i,2}-t_{i,1}$	1	$\frac{3-\beta}{2}$	$2-\beta$	$\frac{4-2\beta}{3-\beta}$
$t_{i,r} > t_{i,2}-t_{i,1}$	$\hat{t}_{i,w} = 0$	1	$\frac{3-\beta}{2}$	$\frac{2}{1+\beta}$	$\frac{4}{(3-\beta)(1+\beta)}$
$\alpha^* = \frac{3-\beta}{2}, \beta = \max_i \frac{t_{i,1}}{t_{i,2}}$					

5 Conclusions

The Recoverable Canadian Traveller Problem is valuable and important for the traffic congestion problems. Most previous studies are based on classical competitive analysis. The classical competitive analysis is the most fundamental and important framework to study online problems, but it is not very flexible. In this paper, we present an optimal online strategy - a comparison strategy and prove its competitive ratio for Recoverable-CTP in a special network. Moreover, with the different forecasts of the recovery time, some online strategies are given under the risk-reward framework, and the rewards and the risks of different strategies are analysed. For the Recoverable Canadian Traveller Problem, there are some further directions, such as, how to deal with the problems in general networks under the risk-reward framework.

References

1. Papadimitriou, C.H., Yannakakis, M.: Shortest paths without a map. Theoretical Computer Science 84(1), 127–150 (1991)
2. Bar-Noy, A., Schieber, B.: The Canadian traveller problem. In: Proceedings of the second annual ACM-SIAM Symposium on Discrete Algorithms, pp. 261–270 (1991)

3. David, S.B., Borodin, A.: A new measure for the study of the on-line algorithm. Algorithmica 11, 73–91 (1994)
4. Westphal, S.: A note on the *k*-Canadian traveller problem. Information Processing Letters 106(3), 87–89 (2008)
5. Xu, Y.F., Hu, M.L., Su, B., Zhu, B.H., Zhu, Z.J.: The Canadian Traveller Problem and Its Competitive Analysis. Journal of Combinatorial Optimization, 4 (in press, 2008)
6. Su, B., Xu, Y.F.: Online recoverable Canadian traveller problem. In: Proceedings of the International Conference on Management Science and Engineering, pp. 633–639 (2004)
7. Sleator, D., Tarjan, R.: Amortized efficiency of list update and paging rules. Communications of the ACM 28(2), 202–208 (1985)
8. Borodin, A., El-Yaniv, R.: Online computation and competitive analysis. Cambridge University Press, Cambridge (1998)
9. Fiat, A., Rabani, Y., Ravid, Y.: Competitive *k*-server algorithms. In: Proceedings of the 22nd IEEE Symposium on Foundation of Computer Science, pp. 454–463 (1990)
10. Fiat, A., Woeginger, G.J.: Online algorithms: The state of art. Springer, Heidelberg (1998)
11. Al-Binali, S.: A risk-reward framework for the competitive analysis of financial games. Algorithmica 25, 99–115 (1999)