Competitive Algorithms for Online Leasing Problem in Probabilistic Environments

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Abstract. We integrate probability distribution into pure competitive analysis to improve the performance measure of competitive analysis, since input sequences of the leasing problem have simple structure and favorably statistical property. Let input structures be the characteristic of geometric distribution, and we obtain optimal on-line algorithms and their competitive ratios. Moreover, the introducing of interest rate would diminish the uncertainty involved in the process of decision making and put off the optimal purchasing date.

1 Introduction

In recent years, competitive analysis has been gaining recognition for being a complementary approach in the analysis of algorithmic decision-making under uncertainty. With the emergence of many on-line financial papers, the competitive approach is shown to be productive for a variety of financial problems. For on-line leasing problem, the prototype was the well-known "Ski-Rental" example put forward by Karp in the field of theoretical computer science in 1992 [9]. Subsequently, a series of research has been carried out on the basic model. In 1994, Karlin et al. made on-line analysis for what they called "the ski-rental-family of problem" [8]. In 1998, S. Irani et al. studied the situation which the purchasing price varies but the rental cost stays at a fixed price [7]. In 1999, R. El-Yaniv et al. investigated the leasing problem with interest rate [4]. In the same year, S. al-Binali developed a famous Risk–Reward Framework to analyze the rental problem and unidirectional trading problem [2]. In 2001, S. Albers et al. introduced and explored natural delayed information and action models to investigate several well-known on-line problems inclusive of the rental problem [1]. However, the previous research always avoids probabilistic assumption intentionally. In 2002, H. Fujiwara et al. firstly integrated probabilistic distribution into pure competitive analysis to study on-line leasing problem, while they suppose that input sequences are drawn form the exponential distribution [6].

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2 Our Contributions

The purpose of this study is to improve the performance measure of competitive analysis by integrating distribution information into pure competitive analysis for the leasing problem. Similar to the results in [6], we also obtain several interesting results as follows. If the average cost of always leasing is less than the purchasing cost, the optimal strategy for an investor is to lease the equipment forever. Otherwise, the optimal strategy is to purchase the equipment after leasing several periods, which the optimal purchasing data would be determined by using the dichotomous search algorithm in the polynomial time. Furthermore, we introduce the nominal interest rate on the market into the model. It could be found that the introducing of interest rate would diminish uncertainty involved in the process of decision making and put off the purchasing date. Although this is only one step toward a more realistic solution of the problem, the introduction of this parameter considerably complicates the analysis and also arises some new issues that do not exist in the Fujiwara's model with no interest rate.

Moveover, the reasons based on such consideration are as follows. Pure competitive analysis always assumes that an investor has no information for input sequences. Indeed, whenever a decision-maker does have some side information or partial (statistical) knowledge on the evolution of input sequences it would be a terrible waste to ignore it, which is precisely what the competitive ratio does. In this case the use of competitive algorithms may lead to inferior performance relative to Bayesian algorithms. Moreover, it is hardly true that competitive analysis of the worst case intentionally emphasizes on difficulty to estimate the input distribution. Other than many combinatorial problems with more complicated input structures, there do exist a number of interesting problems with relatively simple and tractable input structures. We can characterize accurately their input structures by using statistical theory. Hence, stochastic competitive analysis in this paper, as well as in [6], might help to overcome these difficulties.

The reason that we consider the on-line rental problem with geometric distribution comes from the literature [10] which analyzes a class of optimal stopping of geometric distribution and from the "tossing coin" idea that the leasing doesn't cease until the purchasing appears. However, the literature [6] highlights the continuous model with the exponential distribution, while we study a discrete model that assumes the evolution of input sequences subject to the characteristic of geometric distribution. Maybe, we could also amend several aspects of the literature [6] as follows. (i) The continuous model could be unnecessarily equivalent to the discrete case because the leasing problem is essentially discrete; (ii) Geometric distribution may be more reasonable than exponential distribution to depict input structures of the leasing problem, because the leasing activity every period is similar to doing the Bernoulli trial what on earth to rent continuously or to purchase immediately; (iii) In our discrete model, the immediately purchasing at the beginning, i.e. the strategy $A(0)$, could be also an optimal strategy in practice, and the competitive ratio $C(0)$ is a finite value, while it could be not the case that the literature [6] pointed out the competitive ratio $c(k)$ diverging to $+\infty$ as k approaching to zero.

3 Optimal Analysis of Online Algorithms

For on-line problem, we consider the following deterministic on-line strategies $A(k)$ $(k = 0, 1, 2, \cdots)$: rent up to k times and then buy. Thus let $Cost_{ON}$ be the optimal cost of on-line algorithms, and let $Cost_{OPT}$ be the optimal cost of off-line algorithms. In this paper, we consider the discrete model. Therefore the conception of the stochastic competitive ratio is defined as follows.

Definition 1. Let the number of the leasing be a stochastic variable X which is subject to some type of probability distribution, and the probability function is $P(X = t)$, where t is the number of actual leasing. Then the discrete stochastic competitive ratio is defined as

$$C(k) = E_X \frac{Cost_{ON}(X, k)}{Cost_{OPT}(X)} = \sum_{t=0}^{\infty} \frac{Cost_{ON}(t, k)}{Cost_{OPT}(t)} P(X = t), \tag{1}$$

where $P(X = t)$ is a probability function that investors approximately estimate for input structures. We consider that the inputs are drawn from the geometric distribution, and let the hazard rate of continuous renting in every period activity be θ, and then the probability function is $P(X = t) = (1 - \theta)\theta^{t-1}$ $(t = 0, 1, 2, 3, \cdots)$.

Note that there is an essential difference of definitions between the stochastic competitive ratio in this paper and the randomized competitive ratio in the literature [9], [4]. The former indicates that input structure may be subject to some probability distribution, i.e. investor has certain information distribution, while the latter means that on-line players or adversary players choose some strategy randomly in the strategy space set. Thereby we use different terms to imply different meanings.

3.1 Leasing in a Market Without Interest Rate

Let the costs of renting and purchasing equipment be 1 and positive integer s, respectively. Obviously, optimal off-line decision-making cost is

$$Cost_{OPT}(t) = \min\{s, t\}. \tag{2}$$

Based on the strategy set $A(k)$, on-line decision-making cost is

$$Cost_{ON}(t, k) = \begin{cases} t & t \le k, \\ k + s & t > k. \end{cases} \tag{3}$$

Obviously, the optimal strategy is immediately purchasing if s were equal to 1, so s is at least 2. We also make the assumption that the player needs the equipment throughout n contiguous time periods.

According to (1), (2), and (3), we could obtain that, for k=0, 1, 2, 3, $\cdots$, s,

$$C(k) = (1 - \theta^k) + (k + s)(1 - \theta) \sum_{t=k+1}^{s} \frac{1}{t}\theta^{t-1} + \frac{k + s}{s}\theta^s, \tag{4}$$

and for $k = s+1, s+2, s+3, \cdots$,

$$C(k) = (1-\theta^s) + \frac{1-\theta}{s} \sum_{t=s+1}^{k} t\theta^{t-1} + \frac{k+s}{s}\theta^k. \quad (5)$$

Then we obtain the following result by theoretical analysis.

Theorem 1. 1). If $\frac{1}{1-\theta} < s$, then the average cost of always leasing is less than the purchasing cost s. The optimal strategy for an investor is to lease the equipment forever, and the competitive ratio is $1 + \frac{\theta^s}{s(1-\theta)}$;

2). If $\frac{1}{1-\theta} = s$, then the average cost of always leasing is equal to the purchasing cost s. The optimal strategy for an investor is to purchase the equipment after leasing $s-1$ periods, and the competitive ratio is $1 + (1-\frac{1}{s})^s$;

3). If $\frac{1}{1-\theta} > s$, then the average cost of always leasing is greater than the purchasing cost s. The optimal strategy for an investor is to purchase the equipment after leasing k_0 periods, and the competitive ratio is $1-[1-\frac{k_0 s(1-\theta)}{k_0+1} - \frac{s^2(1-\theta)}{k_0+1}]\theta^{k_0}$, where k_0 satisfies $(1-\theta)s^2 - 0.09s - 1 < k_0 < (1-\theta)s^2 - 1$, which the decision-making data k_0 must be determined by using the dichotomous search algorithm in the polynomial time $O(\log s)$;

4). If $\frac{1}{1-\theta} \to \infty$, then the average cost of always leasing limits to ∞, and the optimal competitive ratio of any strategy $A(k)$ is $1 + \frac{k}{s}$. The optimal strategy for an investor is to purchase the equipment at the very beginning, and the competitive ratio approaches to 1.

Whichever case to consider, the competitive ratio that the investor takes the strategy $A(s-1)$, even when there is a large deviation for the hazard ratio θ to be estimated, also is better than the deterministic competitive ratio $2 - \frac{1}{s}$ in [9], and the randomized competitive ratio in [4]. For example, if $s = 10$ and $\theta = 0.95$, then the competitive ratio 1.56722 in this paper is better than the competitive ratio 1.9 in [9], and better than the randomized competitive ratio 1.582 in [4].

3.2 Leasing in a Market with Interest Rate

When considering alternative financial decisions, an agent must consider their net present value, that is, accounting for the market interest rate is an essential feature of any reasonable financial model. Hence, let i be the nominal interest rate in the financial market. Without loss of generality we assume that $\frac{1}{s} > \frac{i}{1+i}$. This is a reasonable assumption for any practical use because the purchase price of the equipment must be less than the present discount value of the alternative of always leasing ($s < \sum_{j=0}^{\infty} \frac{1}{(1+i)^j}$). Otherwise, the online player can attain a competitive ratio of 1 by simply never purchasing the equipment. Set $\beta = \frac{1}{1+i}$, and then $\frac{1}{s} + \beta - 1 > 0$. From economic point of view, $\frac{1}{s} + \beta - 1$ is relative opportunity cost to purchase equipment. In addition, let $\triangle = (\frac{1}{s} + \beta - 1)^{-1}$.

Clearly, the adversary player will never purchase the equipment after leasing it for some time (as in [4]). Therefore, for any n optimal offline decision-making cost is

$$Cost_{OPT}(t) = \begin{cases} \frac{1-\beta^t}{1-\beta} & t \leq n^*, \\ s & t > n^*, \end{cases} \tag{6}$$

where n^* is the number of rentals whose total present value is s. In other words, n^* is the root of $\frac{1-\beta^n}{1-\beta} = s$. That is $n^* = \frac{\ln(1-s(1-\beta))}{\ln \beta}$. Based on the strategy set $A(k)$, on-line decision-making cost is

$$Cost_{ON}(t,k) = \begin{cases} \frac{1-\beta^t}{1-\beta} & t \leq k, \\ s\beta^k + \frac{1-\beta^k}{1-\beta} & t > k. \end{cases} \tag{7}$$

According to (1), (6), and (7), we could obtain that, for k=0, 1, 2, 3, $\cdots$, n^*,

$$C(k) = (1-\theta^k)+(1-\theta)(1-\beta^{n^*+k}) \sum_{t=k+1}^{n^*} \frac{1}{1-\beta^t}\theta^{t-1} + (\beta^k + \frac{1-\beta^k}{s(1-\beta)})\theta^{n^*}, \tag{8}$$

and for $k = n^* + 1, n^* + 2, n^* + 3, \cdots$,

$$C(k) = (1-\theta^{n^*}) + \frac{1-\theta}{s(1-\beta)} \sum_{t=n^*+1}^{k} (1-\beta^t)\theta^{t-1} + (\beta^k + \frac{1-\beta^k}{s(1-\beta)})\theta^k. \tag{9}$$

Then we obtain the following result by theoretical analysis.

Theorem 2. 1). If $\frac{1}{1-\theta} < \frac{\triangle}{1+i}$, then the average cost of always leasing without interest rate is less than the present discount value of the reciprocal of relative opportunity cost. The optimal strategy for an investor is to lease the equipment forever, and the competitive ratio is $1 + \frac{1+\beta(1-2\theta)}{\triangle(1-\beta)(1-\beta\theta)}\theta^{n^*}$;

2). If $\frac{1}{1-\theta} = \frac{\triangle}{1+i}$, then the average cost of always leasing without interest rate is equal to the present discount value of the reciprocal of relative opportunity cost. The optimal strategy for an investor is to buy the equipment after $n^* - 1$ periods, and the competitive ratio is $1 + (\frac{\theta}{1+i})^{n^*}$;

3). If $\frac{1}{1-\theta} > \frac{\triangle}{1+i}$, then the average cost of always leasing without interest rate is greater than the present discount value of the reciprocal of relative opportunity cost. The optimal strategy for an investor is to buy the equipment after k_0 periods, and the competitive ratio is $1 + \frac{s\beta(1-\theta)(1-\beta^{n^*+k_0}) - \beta^{n^*}(1-\beta^{k_0+1})}{\beta^{n^*}(1-\beta^{k_0+1})}\theta^{k_0}$, where the decision-making data k_0 is established by using the dichotomous search algorithm in the polynomial time $O(\log n^*)$;

4). If $\frac{1}{1-\theta} \to \infty$, i.e. the average cost of always leasing without interest rate limits to $+\infty$, then the optimal competitive ratio of any strategy $A(k)$ is $\frac{1}{s(1-\beta)} + (1 - \frac{1}{s(1-\beta)})\beta^k$. The optimal strategy for an investor is to purchase the equipment at the very beginning, and the competitive ratio approaches to 1.

Note that Theorem 2 is the further extension of Theorem 1. If $i \to 0$, then $n^* \to s$, and $\frac{\triangle}{1+i} \to s$. Similar to Theorem 1, if $s = 10$, and $\theta = 0.95$, and $i = 0.01$, then the competitive ratio of non-optimal strategy $A(9)$ that is 1.4983 is better than the competitive ratio 1.9 in [9], and better than the randomized competitive ratio 1.582 in [4], and better than the stochastic competitive ratio

1.56722 without interest rate. Moreover, it could be found that the entrance of interest rate diminishes uncertainty involved in financial decision making and puts off optimal purchasing date. For example, if $s = 19$ and $\theta = 0.98$, the introducing of interest rate $i = 0.02$ results in that the competitive ratio reduces, and that the optimal strategy $A(6)$ with no interest rate is postponed to become the optimal strategy $A(11)$ with interest rate.

4 Concluding Remarks

Although Ran El-Yaniv proposed the axiom set of the competitive ratio, the conception has still inherent and insurmountable limitations [3]. It is still the subject of worthy attention how to improve the performance measure of the competitive ratio by combining other methods. Furthermore, it also is the subject of research how to depict input information under uncertainty, while we think that the theory of Rough set and possibility distribution may be an useful tool to be integrated into pure competitive analysis to improve the performance measure of on-line algorithms in the future.

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