# The On-Line Rental Problem with Risk and Probabilistic Forecast*

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**Abstract.** This paper proposes a generalized on-line risk-reward model, by introducing the notion of the probabilistic forecast. Using this model, we investigate the on-line rental problem. We design the risk rental algorithms under the basic probability forecast and the geometric distribution probability forecast, respectively. In contrast to the existing competitive analyses of the on-line rental problem, our results are more flexible and can help the investor choosing the optimal algorithm according to his/her own risk tolerance level and probabilistic forecast. Moreover, we also show that this model has a good linkage to the stochastic competitive ratio analysis.

## 1 Introduction

In Karp's ski-rental problem [1], we wish to acquire equipment for skiing. However, since we do not know how many times we will use this equipment, we do not know if it is cheaper to rent or to buy. Let $c$ be the rental price every time, and $p$ the buying price. For simplicity, assume that $c \mid p$ and $c, p > 0$. It is easy to prove that the algorithm that achieves the optimal competitive ratio, $2 - c/p$, is to rent for the first $p/c - 1$ times, and then buy the equipment in the $p/c$ time.

Considering the real-life situation of the rental problem, many researchers expanded Karp's ski-rental problem. Irani and Ramanathan [2] studied the rental algorithm under the condition that the buying price is fluctuated but the rental price remains unchanged. Xu [3] further discussed the rental algorithm under the circumstance that the buying price and rental price both fluctuate. EI-Yaniv et al [4] introduced the interest rate factors to the on-line rental problem. Xu [3] considered the discount factors in the on-line rental study. Some more complicated versions based on the ski-rental problem also have been presented, such as EI-Yaniv and Karp's replacement problem [5] and Fleischer's Bahncard problem [6].

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Moreover, some researchers also focus on constructing more flexible competitive analysis frameworks to study on-line rental algorithm. al-Binali [7] proposed the notable on-line risk-reward model. Fujiwara and Iwama [8], and Xu et al [9] integrated probability distribution into the classical competitive analysis [10] to study the rental problem. The main purpose of this paper is to propose a generalized on-line risk-reward model under the probability forecast. We also show a good linkage of our model to the existing competitive analysis frameworks. The rest of this paper is organized as follows. In Section 2, we proposes a generalized on-line risk-reward model, by introducing the notion of the probabilistic forecast. In Section 3, we design the risk rental algorithm under the basic probability forecast. In Section 4, we study the risk rental algorithm under the geometric distribution probability forecast. Concluding remarks and future research are included in Section 5.

## 2 On-Line Risk-Reward Model Under Probabilistic Forecast

In 1985, Sleator and Tarjian [10] proposed the concept of the competitive ratio to study on-line problems, by comparing the performance of on-line algorithms to a benchmark (optimal off-line) algorithm. During this classical competitive analysis, there are an algorithm set $S$ for the on-line decision-maker and a uncertain information set $I$ dominated by the off-line opponent. The on-line decision-maker's goal is to design a good algorithm $A \in S$ to deal with the uncertainty input sequence $\sigma \in I$ of the off-line rival. For a known sequence $\sigma$, let $C_{opt}(\sigma)$ be the total cost of the optimal off-line algorithm to complete $\sigma$. For an on-line algorithm $A$, if there are constants $\lambda_A$ and $\zeta$ satisfying

$$C_A(\sigma) \leq \lambda_A C_{opt}(\sigma) + \zeta$$

for any $\sigma \in I$, then $A$ is called a $\lambda_A$-competitive algorithm and $\lambda_A$ is called the competitive ratio of $A$, where $C_A(\sigma)$ is the total cost taken with algorithm $A$ to complete $\sigma$. That is to say, $\lambda_A = \underset{\sigma \in I}{sup} \frac{C_A(\sigma)}{C_{opt}(\sigma)}$. We denote $\lambda^* = \underset{A \in S}{inf}(\lambda_A)$ as the optimal competitive ratio for the on-line problem. If $\lambda_{A^*} = \lambda^*$, then $A^*$ is called the optimal on-line algorithm.

The above competitive analysis is the most fundamental and significant approach, yet it is not very flexible, especially in the economic management issues, many investors want to manage their risk. Al-Binali [9] first defined the concepts of risk and reward for on-line financial problems. Al-Binali defined the risk of an algorithm $A$ to be $r_A = \frac{\lambda_A}{\lambda^*}$. The greater the value of $r_A$, the higher the risk of $A$. Let $F \subset I$ be a forecast, then denote $\overline{\lambda_A} = \underset{\sigma \in F}{sup} \frac{C_A(\sigma)}{C_{opt}(\sigma)}$ as the restricted competitive ratio of $A$ restricted to cases when the forecast is correct. The optimal restricted competitive ratio under the forecast $F$ is $\overline{\lambda^*} = \underset{A \in S}{inf}(\overline{\lambda_A})$. When the forecast is correct, Al-Binali defined the reward of the algorithm $A$ to be $f_A = \frac{\overline{\lambda^*}}{\overline{\lambda_A}}$.

The above reward definition is based on the certain forecast that is described to be a subset of $I$. When the forecast selected is correct, it will bring reward; otherwise bring risk. This paper extends the certain forecast to the probability forecast. Let $F_1, F_2, ..., F_m$ be a group of subsets of $I$, where $\bigcup F_i = I$ and $F_i \bigcap F_j = \phi$ for $i \neq j$. Denote $P_i$ as the probability that the on-line decision maker anticipates that $\sigma \in F_i$, where $\sum_{i=1}^{m} P_i = 1$. We call $\{(F_i, P_i)|i = 1, 2, ..., m\}$ a probability forecast. Let $\overline{\lambda_{A,i}} = \underset{\sigma \in F_i}{sup} \frac{C_A(\sigma)}{C_{opt}(\sigma)}$ be the restricted competitive ratio under the forecast $F_i$. Let $R_{A,i} = \frac{\lambda^*}{\overline{\lambda_{A,i}}}$ be the reward after the success of the forecast $F_i$. Based on this, we define $\widetilde{\lambda_A} = \sum_{i=1}^{m} P_i \overline{\lambda_{A,i}}$ as the restricted competitive ratio under the probability forecast $\{(F_i, P_i)|i = 1, 2, ..., m\}$, and define $\widetilde{R_A} = \frac{\lambda^*}{\widetilde{\lambda_A}}$ as the reward under the probability forecast.

The reward definition based on the probability forecast has some desired properties.

**Theorem 1.** For any $A \in S$, $\min_i\{R_{A,i}\} \leq \widetilde{R_A} \leq \max_i\{R_{A,i}\}$.

**Proof.** Since $\widetilde{\lambda_A} = \sum_{i=1}^{m} P_i \overline{\lambda_{A,i}}$, $\min_i\{\overline{\lambda_{A,i}}\} \leq \widetilde{\lambda_A} \leq \max_i\{\overline{\lambda_{A,i}}\}$. Consequently, $\min_i\{\lambda^*/\overline{\lambda_{A,i}}\} \leq \lambda^*/\widetilde{\lambda_A} \leq \max_i\{\lambda^*/\overline{\lambda_{A,i}}\}$, that is $\min_i\{R_{A,i}\} \leq \widetilde{R_A} \leq \max_i\{R_{A,i}\}$.

Let $\{(F_i, P_i)|i = 1, 2, ..., m\}$ be a probability forecast. We divide $F_i$ into $F_{i,1}$ and $F_{i,2}$, where $F_{i,1} \cup F_{i,2} = F_i$ and $F_{i,1} \bigcap F_{i,2} = \phi$. We also divide $P_i$ into $P_{i,1}$ and $P_{i,2}$, where $P_{i,1} + P_{i,2} = P_i$. In this way, we can construct a more detailed probability forecast based on $\{(F_i, P_i)|i = 1, 2, ..., m\}$, that is $\{(F_1, P_1), (F_2, P_2), ..., (F_{i-1}, P_{i-1}), (F_{i,1}, P_{i,1}), (F_{i,2}, P_{i,2}), (F_{i+1}, P_{i+1}) ..., (F_m, P_m)\}$. Denote $\widetilde{\widetilde{R_A}} = \frac{\lambda^*}{\widetilde{\widetilde{\lambda_A}}}$ as the reward under the newly constructed probability forecast.

**Theorem 2.** For any $A \in S$, $\widetilde{R_A} \leq \widetilde{\widetilde{R_A}}$.

**Proof.** From the definition of the restricted competitive ratio, we know that $\overline{\lambda_{A,i1}} \leq \overline{\lambda_{A,i}}$ and $\overline{\lambda_{A,i2}} \leq \overline{\lambda_{A,i}}$. Besides $P_{i,1} + P_{i,2} = P_i$, thus $\frac{\lambda^*}{\widetilde{R_A}} - \frac{\lambda^*}{\widetilde{\widetilde{R_A}}} = P_i \overline{\lambda_{A,i}} - P_{i,1} \overline{\lambda_{A,i1}} - P_{i,2} \overline{\lambda_{A,i2}} \geq 0$, that is $\widetilde{R_A} \leq \widetilde{\widetilde{R_A}}$.

Theorem 2 shows that if a probability forecast can be described more detailedly, the reward under the probability forecast will be greater.

Based on these newly introduced concepts, we propose a generalized risk-reward model under the probability forecast. If $r$ is the risk tolerance level of the on-line decision maker (where $r \geq 1$ and higher values of $r$ denote a higher risk tolerance), then denote $S_r = \{A|\lambda_A \leq r\lambda^*\}$ by the set of all algorithms with the risk tolerance level $r$. Our main aim is to look for an optimal risk algorithm $A^* \in S_r$ that maximizes the reward under the probability forecast $\{(F_i, P_i)|i = 1, 2, ..., m\}$, that is $\widetilde{R_{A^*}} = \underset{A \in S_r}{sup} \frac{\lambda^*}{\widetilde{\lambda_A}}$. The mathematic model can be described as follows:

$$\begin{cases} \max\limits_{A} \ \ \widetilde{R_A} = \frac{\lambda^*}{\widetilde{\lambda_A}} \\ s.t \ \ \lambda_A \leq r\lambda^* \end{cases}. \tag{1}$$

The steps to use this model can be described as follows.
**Step 1:** Divide $I$ into $F_1, F_2, ..., F_m$, where $\bigcup F_i = I$ and $F_i \bigcap F_j = \phi$ for $i \neq j$;
**Step 2:** Denote $P_i$ as the probability that the on-line decision maker anticipates that $\sigma \in F_i$, where $\sum_{i=1}^{m} P_i = 1$;
**Step 3:** According to definitions, compute the risk and reward under the probability forecast $\{(F_i, P_i)|i = 1, 2, ..., m\}$;
**Step 4:** Set the risk tolerance level to be $r$;
**Step 5:** Solve the model (1) to obtain the optimal risk algorithm $A^*$.

In the following two sections, we will demonstrate the model that we have just introduced using Karp's ski-rental problem.

## 3 Risk Rental Algorithm Under Basic Probability Forecast

We consider the following deterministic on-line rental algorithm $T$: rent up to $T-1$ times and buy in $T$. Let $Cost_T(t)$ and $Cost_{opt}(t)$ denote the cost of the on-line algorithm $T$ and the cost of the optimal off-line algorithm, respectively, where $t$ is the total number of the actual leases.

For the off-line rental problem, if $t \geq p/c$, then buy; otherwise rent. So we have that

$$Cost_{opt}\{t\} = \begin{cases} ct & 0 \leq t < p/c \\ p & p/c \leq t \end{cases}. \tag{2}$$

For the on-line problem, if $t < T$, then always lease. According to on-line algorithm $T$ $(T = 0, 1, 2, ...)$, then it is not difficult to see that

$$Cost_T\{t\} = \begin{cases} ct & 0 \leq t \leq T \\ cT + p & T < t \end{cases}. \tag{3}$$

According to the off-line optimal rental algorithm, we construct a basic probability forecast, $\{(F_1, P_1), (F_2, P_2)\}$, as follows.

**Forecast** $F_1$: $F_1 = \{t : t < p/c\}$. The probability when $F_1$ appears is $P_1$.
**Forecast** $F_2$: $F_2 = \{t : t \geq p/c\}$. The probability when $F_2$ appears is $P_2$.

**Theorem 3.** When setting the risk tolerance level $r \geq \frac{2p}{2p-c}$, the optimal risk rental algorithm under the probability forecast $\{(F_1, P_1), (F_2, P_2)\}$ is

$$T^* = \begin{cases} p/c, & P_1 > 1/2 \\ \frac{p}{c}\sqrt{\frac{P_1}{1-P_1}}, & p^2/((2pr - cr - p)^2 + p^2) \leq P_1 \leq 1/2 \\ p^2/(2prc - c^2 r - pc), & P_1 \leq p^2/((2pr - cr - p)^2 + p^2) \end{cases}; \tag{4}$$

otherwise, when $1 \leq r \leq \frac{2p}{2p-c}$,

$$T^* = \begin{cases} p/c, & P_1 > 1/2 \\ p^2/(2prc - c^2 r - pc), & P_1 \leq 1/2 \end{cases}. \tag{5}$$

**Proof.** According to the definition of the competitive ratio, we know that $\lambda_T = \frac{cT+p}{min\{cT,p\}}$. Since the risk tolerance level set is $r$, we have that $\lambda_T \leq r(2-c/p)$. When $cT \leq p$, we have that $\lambda_T = \frac{cT+p}{cT} \leq r(2-c/p)$, that is $T \geq \frac{p^2}{2prc-c^2r-pc}$. When $cT > p$, we have that $\lambda_T = \frac{cT+p}{p} \leq r(2-c/p)$, that is $T \leq \frac{2pr-cr-p}{c}$. Consequently,

$$\frac{p^2}{2prc-c^2r-pc} \leq T \leq \frac{2pr-cr-p}{c} \tag{6}$$

Denote $S_r = [\frac{p^2}{2prc-c^2r-pc}, \frac{2pr-cr-p}{c}]$ as the algorithm set with risk level $r$. From the definition of the restricted competitive ratio under the probability forecast, we have that $\widetilde{\lambda_T} = \sum_{i=1}^{2} P_i \overline{\lambda_{T,i}}$, where $\overline{\lambda_{T,i}} = \sup_{\sigma \in F_i} \frac{C_T(\sigma)}{C_{opt}(\sigma)}$. Consequently,

$$\widetilde{\lambda_T} = \begin{cases} P_1 \frac{cT+p}{cT} + \frac{cT+p}{p}(1-P_1), & T < p/c \\ P_1 + \frac{cT+p}{p}(1-P_1), & T \geq p/c \end{cases}. \tag{7}$$

Solving $\frac{\partial \widetilde{\lambda_T}}{\partial T}$, we find that:

(1) when $P_1 > 1/2$, $\widetilde{\lambda_T}$ is monotony decreasing at $T < p/c$, and monotony increasing at $T \geq p/c$;

(2) when $P_1 \leq 1/2$, $\widetilde{\lambda_T}$ is monotony decreasing at $T < \frac{p}{c}\sqrt{\frac{P_1}{1-P_1}}$, and monotony increasing at $T \geq \frac{p}{c}\sqrt{\frac{P_1}{1-P_1}}$.

From the above monotony properties of $\widetilde{\lambda_T}$ and equations (6), we can look for the optimal risk rental algorithm $T^*$ that makes $\widetilde{\lambda_T}$ minimum, that is equations (4) and (5).

**Corollary 1.** When $P_1 = 1$, $T^* = p/c$; when $P_1 = 0$, $T^* = p^2/(2prc - c^2r - pc)$. Corollary 1 shows that our model is a generalized risk-reward framework, compared with one presented in Al-Binali [7].

## 4 Risk Rental Algorithm Under Geometric Distribution Forecast

By dividing $F_i$ into $F_{i,1}$ and $F_{i,2}$ (where $F_{i,1} \cup F_{i,2} = F_i$ and $F_{i,1} \bigcap F_{i,2} = \phi$), and dividing $P_i$ into $P_{i,1}$ and $P_{i,2}$ (where $P_{i,1} + P_{i,2} = P_i$), we construct a more detailed probability forecast based on $\{(F_i, P_i)|i = 1, 2, ..., m\}$, that is $\{(F_1, P_1), (F_2, P_2), ..., (F_{i-1}, P_{i-1}), (F_{i,1}, P_{i,1}), (F_{i,2}, P_{i,2}), (F_{i+1}, P_{i+1})..., (F_m, P_m)\}$. For the rental problem, we can obtain the probability distribution of $t$, when repeatedly dividing $\{(F_1, P_1), (F_2, P_2)\}$ in the above way. Fujiwara and Iwama [8], and Xu et al [9] integrated probability distribution into the classical competitive analysis, and introduced the concept of the stochastic competitive ratio (Definition 1) for the on-line rental problem.

**Definition 1.** Let the number of leases be a stochastic variable $X$ subject to some type of probability distribution function $P(X = t)$. The discrete stochastic competitive ratio is then defined as

$$\widetilde{\widetilde{\lambda_T}} = E_X \frac{Cost_T(X)}{Cost_{opt}(X)} = \sum_{t=0}^{\infty} \frac{Cost_T(t)}{Cost_{opt}(t)} P(X = t), \tag{8}$$

where $P(X = t)$ is a probability function that is used by the on-line decision maker to approximate the input structures.

**Note.** It is easy to find that the definition of the discrete stochastic competitive ratio is consistency of one of the restricted competitive ratio under the corresponding probability distribution forecast.

For the rental problem, Xu ea tal [9] consider the geometric distribution function $P(X = t) = \theta^{t-1}(1-\theta),\ (t = 0, 1, 2, 3, \cdots)$, where $\theta$ is the hazard rate of continuous leasing in every period, and $1-\theta$ is the hazard rate of immediately purchasing in every period. In this paper, we only discuss the situation that $\frac{1}{1-\theta} < p/c$ to illustrate our model.

Let $s = p/c$. According to equations (2), (3), and (8), we have, for $T$=0, 1, 2, 3, $\cdots$, $s$, that

$$\widetilde{\widetilde{\lambda_T}} = (1-\theta^T) + (T+s)(1-\theta) \sum_{t=T+1}^{s} \frac{\theta^{t-1}}{t} + \frac{T+s}{s}\theta^s, \tag{9}$$

and for $k = s+1, s+2, s+3, \cdots$,

$$\widetilde{\widetilde{\lambda_T}} = (1-\theta^s) + \frac{(1-\theta)}{s} \Sigma_{t=s+1}^{T} t\theta^{t-1} + \frac{T+s}{s}\theta^T. \tag{10}$$

Then we obtain the following result (Theorem 4).

**Theorem 4.** When setting the risk tolerance level to be $r$, the optimal risk algorithm under the geometric distribution forecast is $T^{**} = \frac{2pr-cr-p}{c}$ (we only discuss the situation that $\frac{1}{1-\theta} < s$).

**Proof.** For $t < s-1$, we have that

$$\begin{aligned} \widetilde{\widetilde{\lambda_{T+1}}} - \widetilde{\widetilde{\lambda_T}} &= -\frac{s(1-\theta)}{T+1}\theta^T + \frac{1}{s}\theta^s + (1-\theta)\Sigma_{t=T+1}^{s} \frac{\theta^{t-1}}{t} \\ &\le -\frac{s(1-\theta)}{T+1}\theta^T + \frac{1}{s}\theta^s + \frac{1-\theta}{T+1} \frac{\theta^T - \theta^s}{1-\theta} \\ &= \theta^T\left(\frac{1}{T+1} - \frac{s(1-\theta)}{T+1}\right) + \theta^s\left(\frac{1}{s} - \frac{1}{T+1}\right) < 0 \end{aligned}$$

For $t \ge s-1$, we also have that

$$\widetilde{\widetilde{\lambda_{T+1}}} - \widetilde{\widetilde{\lambda_T}} = \left(\frac{1}{s} - 1 + \theta\right)\theta^T < 0$$

Therefore, we have that $\lambda_{T+1} - \lambda_T < 0$ for any $T$. Besides, because $\widetilde{\widetilde{R_T}} = \frac{\lambda^*}{\widetilde{\widetilde{\lambda_T}}}$ and $T \in S_r$, we have that $T^{**} = \frac{2pr-cr-p}{c}$.

## 5 Conclusions

The classical competitive ratio analysis is the most fundamental and important framework to study online problems. But it is not very flexible, particularly in the financial and investment issues (such as on-line rental problem, on-line currency conversion, on-line auctions problem). Many investors hope to manage the risk. Sometimes for more reward, they are willing to take certain risk. Therefore, the notable online risk-reward idea has been proposed by Al-Binali. However, the existing concept of risk-reward is mainly based on the certainty forecast. In this paper, we further puts forward the online risk- reward model under the probability forecast. The probability forecast will not make the simple judgment about whether the forecast is correct or not, but estimate the probability that the forecast is correct. The newly introduced model makes the risk-reward idea more flexible. Moreover, some researchers presented the concept of the stochastic competitive ratio to improve the performance measure of competitive analysis, by integrating probability distribution into the classical competitive ratio analysis. This paper shows that our model has a good linkage to the stochastic competitive ratio analysis. We also argue that our model is the generalized stochastic competitive ratio analysis. By using Karp's ski-rental problem, we demonstrate the on-line risk-reward model under the probability forecast. In general, it is hard for an on-line decision maker to accurately estimate the probability forecast of the future inputs. Therefore, in our future research, we will explore the on-line risk-reward model in linguistic environments, by introducing the notations and operational laws of the linguistic variables.

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