ON THE MINIMUM DISTANCE DETERMINED BY $n(\leq 7)$ POINTS IN AN ISOSCELE RIGHT TRIANGLE

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Abstract

Let T denote a finite set of points in a unit isoscele right triangle (i.e., the right sides are both one), $f(T)$ the minimum distance between pairs of points of T, and

$$f_{\triangle}(n) = \max_{\|T\|=n} f(T).$$

In this paper, the exact values of $f_{\triangle}(n)$ for $2 \leq n \leq 7$ and the corresponding configurations are given.

Key words. Configuration, unit square, limitation

1. Introduction

The problem of the minimum distance determined by n points in some specific bounded convex regions was studied for a long time, especially the regions are squares or circles. The exact values of the maximum distance determined by n points in a unit square or a unit circle are known only for some small n[1−4].

For a unit square and the cases of $n \geq 10$ and $n \neq 16$, the exact values of the maximum distance are unknown. For a unit circle and the cases of $n \geq 11$, the exact values of the maximum distance are unknown. Some bounds of the exact values of the maximum distance determined by n points in a unit square and a unit circle were given[5,6], and there are some conjectures on "the best" configurations for some small integer n.

In this paper, the maximum distance determined by n points in a unit isoscele right triangle for $2 \leq n \leq 7$ is discussed. The exact values and the corresponding configurations are given for $2 \leq n \leq 7$. From the results obtained in this paper, the relationship of the exact values and the corresponding configurations between a unit square and a unit isoscele right triangle is obvious for $2 \leq n \leq 7$.

Let R be a convex region, S be a finite set of points in R, $f_R(S)$ the minimum distance between pairs of points of S, and

$$f_R(n) = \max_{\|S\|=n} f_R(S).$$

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if R is a unit square, we use $f(n)$ instead of $f_R(n)$. If R is a unit isoscele right triangle, we use $f_\triangle(n)$ instead of $f_R(n)$.

If $f_R(S) = f_R(n)$ with $\|S\|=n$, then we use $M_R(n)$ to denote the corresponding configuration. Similarly we have $M(n)$ for the unit square and $M_\triangle(n)$ for the unit isoscele right triangle.

In the next part of this paper, the following lemma is often used, which is obvious.

Lemma 1.1. If the distance between any two vertices of a closed convex polygon is less than or equal to m, then the distance between the two points lying in or on the polygon can be equal to m only when points are two vertices.

2. For $2\leq k\leq 7$

The values of $f_\triangle(n)$ for $2 \leq n \leq 7$ are given in Table 1 and the corresponding $M_\triangle(n)$ for $2 \leq n \leq 7$ are shown in Fig. 1.

Table 1.

k	$f_\triangle(k)$	
2	$\sqrt{2}$	~1.4142
3	1	=1.0000
4	$\frac{\sqrt{2}}{2}$	~0.7071
5	$2(2-\sqrt{3})$	~0.5359
6	$\frac{1}{2}$	~0.5000
7	$\frac{(4+\sqrt{2})(\sqrt{1+2\sqrt{2}}-\sqrt{2})}{7}$	~0.4195

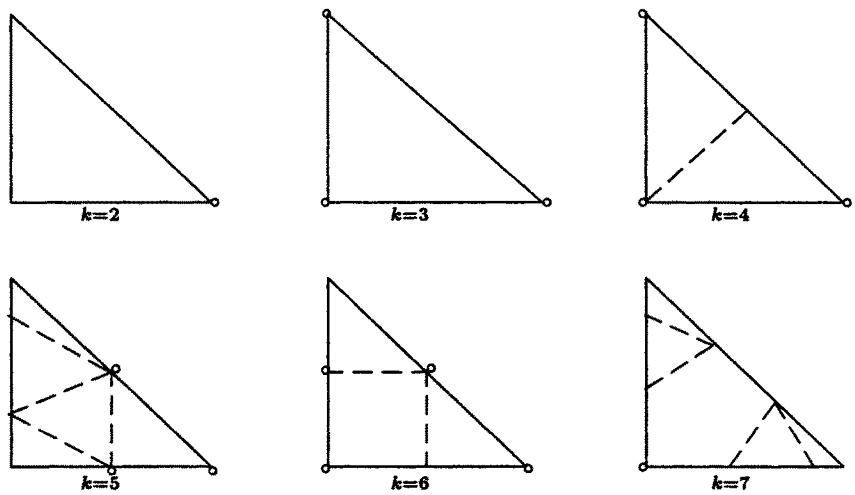


Fig. 1

The cases $k = 2,3$ and 4 can be solved easily. For $k = 5$ and 6, we can solve them similarly to the cases of $f(n)$ for $n = 7$ and 9 as solved by Schaer and Meir[5] and Schaer[3]. From the known result we have

$$f_\triangle(2) = f(2), \quad f_\triangle(3) = f(4), \quad f_\triangle(4) = f(5),$$
$$f_\triangle(5) = f(7), \quad f_\triangle(6) = f(9),$$

and $f_\triangle(7)$ is just the bound for $f(10)$ given by Schaer[4] and independently by Klaus Schluter[6]. The exact value of $f(10)$ is still an unsolved problem.

For $k = 7$, as we shall show here, the best configuration $M_\triangle(7)$ is not difficult to guess. This figuration is just half of the configuration for the best known bound of $f(10)$. We shall show that, for any seven points P_i $(1 \le i \le 7)$ in a closed unit isoscele right triangle,

$$\min_{1\le i<j\le 7} d(p_i, p_j) \le f_\triangle(7),$$

and that the equality holds only for the $M_\triangle(7)$ $\big(d(p_i,p_j)$ denotes the distance between p_i and $p_j\big)$.

Let ϕ be a set of seven points p_i $(i = 1, 2, \cdots, 7)$ in a closed unit isoscele right triangle with

$$\min_{1\le i<j\le 7} d(p_i, p_j) \ge m = \frac{(4+\sqrt{2})(\sqrt{1+\sqrt{2}}-\sqrt{2})}{7}. \tag{1}$$

We shall show that there is just one such set, namely, the one shown in Fig. 1, in which the equality in (1) obviously holds. In the proof we shall use the auxiliary points indicated in Fig. 2.

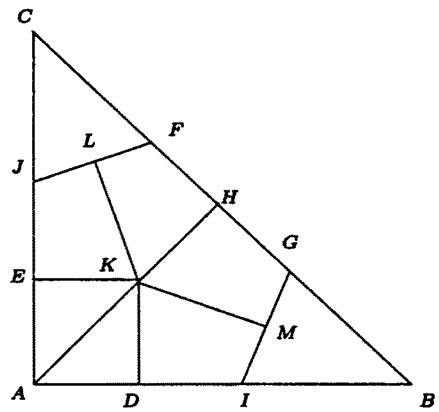


Fig. 2

The points A, B, C are the vertices of a unit isoscele right triangle, H is the midpoint of the edge BC. The points I, G, F, J on the edges are defined by

$$d(B,G) = d(B,I) = d(C,F) = d(C,J) = m.$$

The points M and L are the midpoints of IG and JF. The points A, D, K, E are defined to be the vertices of the square with edge length $\frac{\sqrt{2}}{2}m$, and $d(A,K) = m$.

From this construction, we first want to prove that at most one point belonging to ϕ can be located in each region (Seven Regions).

Proposition 2.1. If (1) holds, then there is at most one point in ϕ that can be located in the quadrilateral $HGMKH$ (or $HFLKH$), and also the same in the quadrilateral $DIMKD$ (or $EJLKE$).

Proof. From Fig. 2 and some numerical calculations, we have

$$d(K,h) = d(H,G) < m,$$
$$d(M,G) < m, \qquad d(M,K) < m,$$
$$d(H,M) < m, \qquad d(K,G) < m.$$

From Lemma 1.1 and the symmetric property of Fig. 2, we know that there is at most one point in ϕ that can be located in the quadrilateral $HGMKH$ (or $HFLKH$).

By a similar argument, we have that there is at most one point in ϕ that can be located in quadrilateral $DIMKD$ (or $EJLKE$).

Proposition 2.2. If (1) holds, then $K \notin \phi$.

Proof. Assume $K \in \phi$. Obviously (see Fig. 2) no points in ϕ can be located in the quadrilaterals $DKMID$, $MGHKM$, $KHFLK$ and $LJEKL$. From Lemma 1.1, there are at most two points in ϕ that can be located in triangle $IGBI$. In square $ADKEA$, A can be in ϕ. So we know that at most six points in a unit isosceles right triangle can be in ϕ, which is in contradiction with the definition of ϕ, since $|\phi| = 7$.

Proposition 2.3. If (1) holds, then no two points in the square $ADKEA$ can be in ϕ.

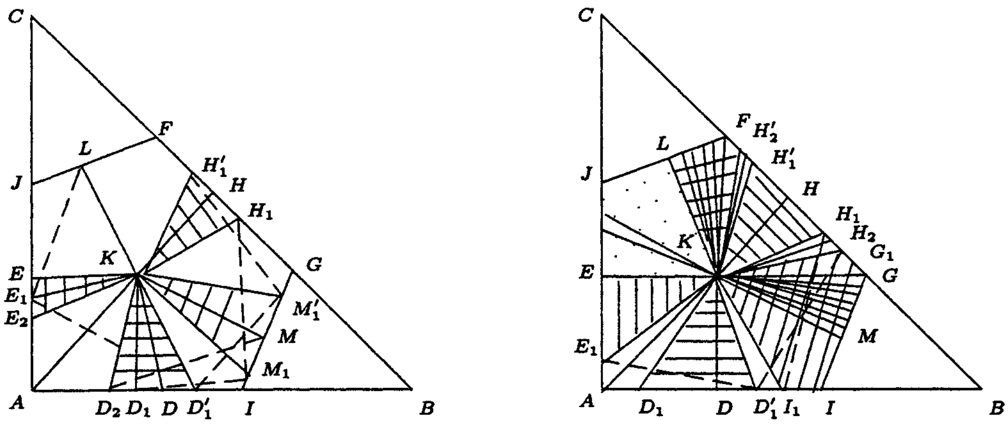
Proof. From Proposition 2.2, we know that $K \notin \phi$. If there are two points in ϕ located in the square $ADKEA$, then from Lemma 1.1, we know that the two points must be D and E. So we have $J, I \notin \phi$, and there are at most four points in ϕ that can be located in the union of the triangles $JCFJ$ and $IBGI$ and the quadrilateral $GMKLFG$, and also there are no other points except E and D in the union of square $EKDAE$ and the quadrilaterals $JLKEJ$ and $KMIDK$. This yields a contradiction to the definition of ϕ, since $|\phi| = 7$.

From Propositions 2.1, 2.2 and 2.3, we know that at most one point in ϕ can be located in each of the regions $ADKEA$, $DIMKD$, $MGHKM$, $HFLKH$, $LJEKL$. From Lemma 1.1, there are at most two points in ϕ that can be located in the triangle $JFCJ$. If there are two points in ϕ that can be located in the triangle $JFCJ$, then they must be C and J (or C and F). In this case, we can assume that J is in the quadrilateral $LJEKL$ (or F in $HFLKH$). Similar argument and assumption can be made for the triangle $BGIB$.

From the above discussion, we have the following lemma.

Lemma 2.4. There must be exactly one point of ϕ located in the regions $ADKEA$, $DKMID$, $BGIB$, $GHKMG$, $HFLKH$, $CJFC$ and $JEKLJ$.

We shall now indicate a procedure by which the location of the points $p_i \in \phi$ may be restricted to the subregion of the preceding one. Iterating this process, the area of the region tends to zero, and at last the points in ϕ are confined to some fixed points. To do this, we need some preconditioning. See Fig. 3 (a).



(a) (b)

Fig. 3

Let D_1 be a point in segment AD and $d(M, D_1) = m$. From Lemma 2.4, we know that there is one point in ϕ that can be located in the region KD_1AED_1 and no points in ϕ can be located inside the triangle D_1KDD_1. By the same method we have E_1 and $d(L, E_1) = m$. So we know that there is one point in ϕ that must be located in the quadrilateral $E_1KD_1AE_1$. Let D_1' be a point in segment DI and $d(D_1', E_1) = m$. We have that there is one point that must be located in the region $D_1'KMID_1'$ and no point in ϕ can be located in the triangle $DKD_1'D$. Let M_1' be a point in segment MG and $d(D_1, M_1') = m$. We have that there is one point in ϕ that must be located in the region $M_1'KHGM_1'$ and no points in ϕ can be located in the triangle $MKM_1'M$. Let H_1' be a point in segment HF and $d(H_1', M_1') = m$. We have no point in ϕ that can be located in the triangle $H_1'KHH_1'$. In the same way, we have H_1, M_1, D_2 and E_2, where $d(H_1, M_1) = m$, $d(M_1, D_2) = m$, $d(H_1', H) = d(H_1, H)$, $d(E, E_2) = d(D, D_2)$. By the same method, we can construct D_2', M_2', H_2', H_2, M_2, D_3, E_3, $D_3', \cdots$. From some numerical calculations we know that for $n \geq 5$, M_n' be on the segment HG. This means that no points in ϕ can be located in the triangle $KGMK$. For the simplicity of proof and without loss of generality, we can construct a point sequence on the segment HG instead of the segment MG. See Fig. 3 (b).

With the method as shown above, let H_1', H_1, I_1, D_1, E_1, D_1', G_1, H_2', H_2 satisfy $d(G, H_1') = d(F, H_1) = d(H_1, I_1) = d(I_1, D_1) = m$, $d(D_1, A) = d(E_1, A)$, $d(E_1, D_1') = d(D_1', G_1) = d(G_1, H_2') = m$, $d(H_2', F) = d(H_2, G)$. And continuing to construct the next point we have I_2, D_2, E_2, D_2', G_2, H_3', H_3, $I_3, \cdots$.

Let

$$
\begin{aligned}
&d(H_i, B) = x_i, \qquad d(I_i, B) = y_i, \qquad d(A, D_i) = z_i,\\
&d(A, D_i') = u_i, \qquad d(B, G_i) = v_i.
\end{aligned}
$$

We have

$$
\begin{cases}
x_1 = \sqrt{2} - 2m,\\
y_1 = \dfrac{\sqrt{2}}{2}x_1 + \sqrt{m^2 - \dfrac{1}{2}x_1^2},\\
z_1 = 1 - y_1 - m,\\
u_1 = \sqrt{m^2 - z_1^2},\\
v_n = \dfrac{\sqrt{2}}{2}(1 - u_1) + \sqrt{m^2 - \dfrac{1}{2}(1 - u_1)^2};
\end{cases}
$$

and for $n > 1$,

$$
\begin{cases}
x_n = \sqrt{2} - m - v_{n-1},\\
y_n = \dfrac{\sqrt{2}}{2}x_n + \sqrt{m^2 - \dfrac{1}{2}x_n^2},\\
z_n = 1 - y_n - m,\\
u_n = \sqrt{m^2 - z_n^2},\\
v_n = \dfrac{\sqrt{2}}{2}(1 - u_n) + \sqrt{m^2 - \dfrac{1}{2}(1 - u_n)^2}.
\end{cases} \tag{2}
$$

Next we shall prove that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ are all convergent. From Lemma 3.4 and the definition of sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ we have

$$
\begin{aligned}
&x_n > v_n, \qquad x_n > m, \qquad v_n > m,\\
&y_i > u_i, \qquad u_i < m, \qquad y_i < 1 - m,\\
&z_i > 0.
\end{aligned}
$$

From numerical calculations, the following inequalities hold:

$$x_2 < x_1, \quad y_2 > y_1, \quad z_2 < z_1,$$
$$u_2 > u_1, \quad v_2 > v_1.$$

Suppose the following inequalities hold for $l \geq 2$:

$$\begin{cases} x_l < x_{l-1}, \\ y_l > y_{l-1}, \\ z_l < z_{l-1}, \\ u_l > u_{l-1}, \\ v_l > v_{l-1}. \end{cases} \tag{3}$$

From (2) we have

$$\begin{aligned} x_{l+1} - x_l &= -v_l + v_{l-1} = -(v_l - v_{l-1}) < 0, \\ y_{l+1} - y_l &= \frac{\sqrt{2}}{2}(x_{l+1} - x_l) + \sqrt{m^2 - \frac{1}{2}x_{l+1}^2} - \sqrt{m^2 - \frac{1}{2}x_l^2} \\ &> \frac{\sqrt{2}}{2}(x_{l+1} - x_l) + \frac{x_l(x_l - x_{l-1})}{2\sqrt{m^2 - \frac{1}{2}x_l^2}} \\ &= (x_l - x_{l+1})\left[\frac{x_l}{2\sqrt{m^2 - \frac{1}{2}x_l^2}} - \frac{\sqrt{2}}{2}\right]. \end{aligned}$$

Since $x_n > m$, we have

$$y_{l+1} - y_l > 0$$

and

$$z_{l+1} - z_l = -(y_{l+1} - y_l) < 0.$$

Similarly, we have

$$u_{l+1} - u_l > 0, \quad v_{l+1} - v_l > 0.$$

From the assumption of induction, we know that (3) holds and the sequence $\{x_n\}$ is a decreasing sequence and has a lower bound m. So the sequence $\{x_n\}$ is a convergent sequence.

Let

$$\lim_{n\to\infty} x_n = x.$$

We have that y, z, u, v satisfy the following equalities:

$$\lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z, \quad \lim_{n\to\infty} u_n = u, \quad \lim_{n\to\infty} v_n = v,$$

and the following equations hold:

$$\begin{cases} x = \sqrt{2} - m - v, \\ y = \frac{\sqrt{2}}{2}x + \sqrt{m^2 - \frac{1}{2}x^2}, \\ z = 1 - y - m, \\ u = \sqrt{m^2 - z^2}, \\ v = \frac{\sqrt{2}}{2}(1-u) + \sqrt{m^2 - \frac{1}{2}(1-u)^2}. \end{cases}$$

To solve the above equations after deleting the extra solutions, we have

$$x = \frac{\sqrt{2}-m}{2}, \qquad y = 1-m, \qquad z = 0,$$
$$u = m, \qquad v = \frac{\sqrt{2}-m}{2}.$$

Since $x = v = \frac{\sqrt{2}-m}{2}$, $y+u=1$ and $z=0$, we know that Fig. 3 (b) can be converted to Fig. 4, where the points E^*, D^*, F^*, G^* are the limit positions of the corresponding point sequences.

As shown in Fig. 4, it is obvious that A, D^*, E^*, G^*, F^* are all in ϕ, and the other two points in ϕ must be located in the shaded areas of the corners B and C. So we have proved that $M_\triangle(7)$ just corresponds to the case of $k=7$ in Fig. 1.

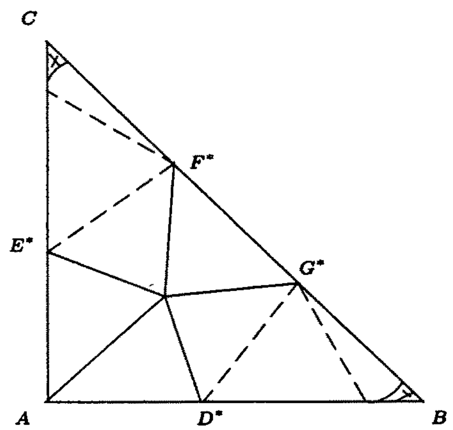


Fig. 4

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